

**Notes on  
Gröbner Bases and Free  
Resolutions of Modules over  
Solvable Polynomial Algebras**

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# Introduction

*Polynomials and power series,  
May they forever rule the world.*

— Shreeram S. Abhyankar

Since the late 1980s, the Gröbner basis theory for commutative polynomial algebras and their modules (cf. [Bu1, 2], [Sch], [BW], [AL2], [Fröb], [KR1, 2]) has been successfully generalized to (noncommutative) solvable polynomial algebras and their modules (cf. [AL1], [Gal], [K-RW], [Kr2], [LW], [Li1], [Lev]). In particular, there have been a noncommutative version of Buchberger’s criterion and a noncommutative version of the Buchberger algorithm for checking and computing

- (1) Gröbner bases of (one-sided) ideals of solvable polynomial algebras, and
- (2) Gröbner bases of submodules of free modules over solvable polynomial algebras,

while the noncommutative version of Buchberger algorithm has been implemented in some well-developed computer algebra systems, such as MODULA-2 [KP] and SINGULAR [DGPS]. By the definition (see Section 1 of [K-RW], or Definition 1.1.3 given in our Chapter 1), a solvable polynomial algebra is a polynomial-like but generally noncommutative algebra. Nowadays it is well known that the class of solvable polynomial algebras covers numerous significant noncommutative algebras such as enveloping algebras of finite dimensional Lie algebras, Weyl algebras (including algebras of partial differential operators with polynomial coefficients over a

field of characteristic 0), more generally a lot of operator algebras, iterated Ore extensions, and quantum (quantized) algebras. So, comparing with the commutative case, the theory of Gröbner bases for solvable polynomial algebras and their modules has created great possibility of studying certain noncommutative algebras and their modules in a “solvable setting” (see [Wik1] for an introduction to “Decision problem” or “Solvable problem”, which may help us to better understand why a “solvable polynomial algebra” deserves its name).

Along the lines in the literature for developing a computational (one-sided) ideal theory and more generally, a computational module theory over solvable polynomial algebras, it seems that a rapid but relatively systematic and concrete introduction to the subject is worthwhile. Thereby, we wrote these lecture notes so as to provide graduates and researchers (who are interested in *noncommutative* computational algebra) with an accessible reference on

- an concise introduction to solvable polynomial algebras and the theory of Gröbner bases for submodules of free modules over solvable polynomial algebras, and
- some details concerning applications of Gröbner bases in constructing finite free resolutions over an arbitrary solvable polynomial algebras, minimal finite  $\mathbb{N}$ -graded free resolutions over an  $\mathbb{N}$ -graded solvable polynomial algebra with the degree-0 homogeneous part being the ground field  $K$ , and minimal finite  $\mathbb{N}$ -filtered free resolutions over an  $\mathbb{N}$ -filtered solvable polynomial algebra (where the  $\mathbb{N}$ -filtration is determined by a positive-degree function).

The first three chapters of these notes grew out of a course of lectures given to graduate students at Hainan University, and the last two chapters are adapted from the author’s recent research work [Li7] (or [Li5] arXiv:1401.5206v2 [math.RA], [Li6] arXiv:1401.5464 [math.RA]). Also, at the level of module theory, these notes may be viewed as supplements of the sections concerning solvable polynomial algebras and their modules in ([Li1], [Li2]).

Throughout the text,  $K$  denotes a field,  $K^* = K - \{0\}$ ;  $\mathbb{N}$  denotes the additive monoid of all nonnegative integers, and  $\mathbb{Z}$  denotes the additive group of all integers; all algebras are associative  $K$ -algebras with the multiplicative identity 1, and modules over an algebra are meant left unitary modules.

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# 1. Solvable Polynomial Algebras

In this chapter we give a concise but easily accessible introduction to solvable polynomial algebras and the theory of left Gröbner bases for left ideals of such algebras. In the first section we introduce the definition of solvable polynomial algebras and some typical examples; also from the definition we highlight some often used properties of solvable polynomial algebras. In Section 2, we introduce left Gröbner bases for left ideals of solvable polynomial algebras via a left division algorithm, and we discuss some basic properties of left Gröbner bases. In Section 3, on the basis of Dickson's lemma we show that every left ideal of a solvable polynomial algebra has a finite Gröbner basis, thereby solvable polynomial algebras are left Noetherian. Since every solvable polynomial algebra has a (two-sided) monomial ordering, a right division algorithm and a theory of right Gröbner bases for right ideals hold true as well, it follows that every solvable polynomial algebra is also right Noetherian. Concerning the algorithmic approach to computing a finite left Gröbner basis, it will be a job of Chapter 2 for modules. In the final Section 4, by employing Gröbner bases of two-sided ideals in free algebras we give a characterization of solvable polynomial algebras, so that such algebras are completely recognizable and constructible in a computational way.

The main references of this chapter are [AL1], [Gal], [K-RW], [Kr2], [LW], [Li1], [Li4], [DGPS], [Ber2], [Mor], [Gr], [Uf].

## 1.1. Definition and Basic Properties

Let  $K$  be a field and let  $A = K[a_1, \dots, a_n]$  be a finitely generated  $K$ -algebra with the set of generators  $\{a_1, \dots, a_n\}$ , that is,  $A$  is an associative ring with the multiplicative identity 1, every element  $a \in A$  is expressed as a finite sum of the form  $a = \sum_i \lambda_i a_{i_1}^{\alpha_1} \cdots a_{i_t}^{\alpha_t}$  with  $\lambda_i \in K$ ,  $a_{i_j} \in \{a_1, \dots, a_n\}$ ,  $\alpha_j \in \mathbb{N}$ ,  $t \geq 1$ , and  $A$  is also a  $K$ -vector space with respect to its additive operation, such that  $\lambda(ab) = (\lambda a)b = a(\lambda b)$  holds for all  $\lambda \in K$  and  $a, b \in A$ . Moreover, we assume that any proper subset of the given generating set  $\{a_1, \dots, a_n\}$  of  $A$  cannot generate  $A$  as a  $K$ -algebra, i.e., the given set of generators is *minimal*.

If, for some permutation  $\tau = i_1 i_2 \cdots i_n$  of  $1, 2, \dots, n$ , the set  $\mathcal{B} = \{a^\alpha = a_{i_1}^{\alpha_1} \cdots a_{i_n}^{\alpha_n} \mid \alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n\}$ , forms a  $K$ -basis of  $A$ , then  $\mathcal{B}$  is referred to as a *PBW  $K$ -basis* of  $A$  (where the phrase “PBW  $K$ -basis” is abbreviated from the well-known *Poincaré-Birkhoff-Witt Theorem* concerning the standard  $K$ -basis of the enveloping algebra of a Lie algebra, e.g., see [Hu], P. 92). It is clear that if  $A$  has a PBW  $K$ -basis, then we can always assume that  $i_1 = 1, \dots, i_n = n$ . Thus, we make the following convention once for all.

**Convention** From now on in this paper, if we say that an algebra  $A$  has the PBW  $K$ -basis  $\mathcal{B}$ , then it means that

$$\mathcal{B} = \{a^\alpha = a_1^{\alpha_1} \cdots a_n^{\alpha_n} \mid \alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n\}.$$

Moreover, adopting the commonly used terminology in computational algebra, elements of  $\mathcal{B}$  are referred to as *monomials* of  $A$ .

### Monomial ordering and admissible system

Suppose that the  $K$ -algebra  $A = K[a_1, \dots, a_n]$  has the PBW  $K$ -basis  $\mathcal{B}$  as presented above and that  $\prec$  is a total ordering on  $\mathcal{B}$ . Then every nonzero element  $f \in A$  has a unique expression

$$f = \lambda_1 a^{\alpha(1)} + \lambda_2 a^{\alpha(2)} + \cdots + \lambda_m a^{\alpha(m)},$$

where  $\lambda_j \in K^*$ ,  $a^{\alpha(j)} = a_1^{\alpha_{1j}} a_2^{\alpha_{2j}} \cdots a_n^{\alpha_{nj}} \in \mathcal{B}$ ,  $1 \leq j \leq m$ .

If  $a^{\alpha(1)} \prec a^{\alpha(2)} \prec \cdots \prec a^{\alpha(m)}$  in the above representation, then the *leading monomial* of  $f$  is defined as  $\mathbf{LM}(f) = a^{\alpha(m)}$ , the *leading coefficient* of  $f$  is defined as  $\mathbf{LC}(f) = \lambda_m$ , and the *leading term* of  $f$  is defined as  $\mathbf{LT}(f) = \lambda_m a^{\alpha(m)}$ .

**1.1.1. Definition** Suppose that the  $K$ -algebra  $A = K[a_1, \dots, a_n]$  has the PBW  $K$ -basis  $\mathcal{B}$ . If  $\prec$  is a total ordering on  $\mathcal{B}$  that satisfies the following three conditions:

- (1)  $\prec$  is a well-ordering (i.e. every nonempty subset of  $\mathcal{B}$  has a minimal element);
- (2) For  $a^\gamma, a^\alpha, a^\beta, a^\eta \in \mathcal{B}$ , if  $a^\gamma \neq 1$ ,  $a^\beta \neq a^\gamma$ , and  $a^\gamma = \mathbf{LM}(a^\alpha a^\beta a^\eta)$ , then  $a^\beta \prec a^\gamma$  (thereby  $1 \prec a^\gamma$  for all  $a^\gamma \neq 1$ ),
- (3) For  $a^\gamma, a^\alpha, a^\beta, a^\eta \in \mathcal{B}$ , if  $a^\alpha \prec a^\beta$ ,  $\mathbf{LM}(a^\gamma a^\alpha a^\eta) \neq 0$ , and  $\mathbf{LM}(a^\gamma a^\beta a^\eta) \notin \{0, 1\}$ , then  $\mathbf{LM}(a^\gamma a^\alpha a^\eta) \prec \mathbf{LM}(a^\gamma a^\beta a^\eta)$ ;

then  $\prec$  is called a *monomial ordering* on  $\mathcal{B}$  (or a monomial ordering on  $A$ ).

If  $\prec$  is a monomial ordering on  $\mathcal{B}$ , then the data  $(\mathcal{B}, \prec)$  is referred to as an *admissible system* of  $A$ .

**Remark.** (i) Definition 1.1.1 above is borrowed from the theory of Gröbner bases for general finitely generated  $K$ -algebras, in which the algebras considered may be noncommutative, may have divisors of zero, and the  $K$ -bases used may not be a PBW basis, but with a (one-sided, two-sided) monomial ordering that, theoretically, enables such algebras have a (one-sided, two-sided) Gröbner basis theory. For more details on this topic, one may refer to ([Li2], Section 1 of Chapter 3 and Section 3 of Chapter 8). Also, to see the essential difference between Definition 1.2.1 and the classical definition of a monomial ordering in the commutative case, one may refer to (Definition 1.4.1 and the proof of Theorem 1.4.6 given in [AL2]).

(ii) The conditions (2) and (3) listed in Definition 1.1.1 mean that  $\prec$  is *two-sided compatible with the multiplication operation of the algebra  $A$* . As one will see soon, that the use of a (two-sided) monomial ordering  $\prec$  on a solvable polynomial algebra  $A$  first guarantees that  $A$  is a domain, and furthermore guarantees an effective (left, right, two-sided) finite Gröbner basis theory for  $A$ .

Note that if a  $K$ -algebra  $A = K[a_1, \dots, a_n]$  has the PBW  $K$ -basis  $\mathcal{B} = \{a^\alpha = a_1^{\alpha_1} \cdots a_n^{\alpha_n} \mid \alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n\}$ , then for any given  $n$ -tuple  $(m_1, \dots, m_n) \in \mathbb{N}^n$ , a *weighted degree function*  $d(\cdot)$  is well defined on nonzero elements of  $A$ , namely, for each  $a^\alpha = a_1^{\alpha_1} \cdots a_n^{\alpha_n} \in \mathcal{B}$ ,  $d(a^\alpha) = m_1 \alpha_1 + \cdots + m_n \alpha_n$ , and for each nonzero  $f = \sum_{i=1}^s \lambda_i a^{\alpha(i)} \in A$  with



$\lambda_i \in K^*$  and  $a^{\alpha(i)} \in \mathcal{B}$ ,

$$d(f) = \max\{d(a^{\alpha(i)}) \mid 1 \leq i \leq s\}.$$

If  $d(a_i) = m_i > 0$  for  $1 \leq i \leq n$ , then  $d(\cdot)$  is referred to as a *positive-degree function* on  $A$ .

**1.1.2. Definition** Let  $d(\cdot)$  be a positive-degree function on  $A$ . If  $\prec$  is a monomial ordering on  $\mathcal{B}$  such that for all  $a^\alpha, a^\beta \in \mathcal{B}$ ,

$$a^\alpha \prec a^\beta \text{ implies } d(a^\alpha) \leq d(a^\beta),$$

then we call  $\prec$  a *graded monomial ordering* with respect to  $d(\cdot)$ .

**Convention** Unless otherwise stated, from now on in this book we always use  $\prec_{gr}$  to denote a graded monomial ordering with respect to a positive-degree function on  $A$ .

As one may see from the literature (or loc. cit) that in both the commutative and noncommutative computational algebra, the most popularly used graded monomial orderings on an algebra  $A = K[a_1, \dots, a_n]$  with the PBW  $K$ -basis  $\mathcal{B}$  are those graded (reverse) lexicographic orderings with respect to the degree function  $d(\cdot)$  such that  $d(a_i) = 1$ ,  $1 \leq i \leq n$ .

### Definition of solvable polynomial algebras and examples

Originally, a (noncommutative) solvable polynomial algebra (or an algebra of solvable type)  $R'$  was defined in [K-RW] by first fixing a monomial ordering  $\prec$  on the standard  $K$ -basis  $\mathcal{B} = \{X_1^{\alpha_1} \cdots X_n^{\alpha_n} \mid \alpha_i \in \mathbb{N}\}$  of the commutative polynomial algebra  $R = K[X_1, \dots, X_n]$  in  $n$  variables  $X_1, \dots, X_n$  over a field  $K$ , and then introducing a new multiplication  $*$  on  $R$ , such that certain axioms ([K-RW], AXIOMS 1.2) are satisfied. In [LW], a solvable polynomial algebra was redefined in the formal language of associative  $K$ -algebras, as follows.

**1.1.3. Definition** If a  $K$ -algebra  $A = K[a_1, \dots, a_n]$  satisfies the following two conditions:

- (S1)  $A$  has the PBW  $K$ -basis  $\mathcal{B}$ ;
- (S2) There is a monomial ordering  $\prec$  on  $\mathcal{B}$ , i.e.,  $(\mathcal{B}, \prec)$  is an admissible

system of  $A$ , such that for all  $a^\alpha = a_1^{\alpha_1} \cdots a_n^{\alpha_n}$ ,  $a^\beta = a_1^{\beta_1} \cdots a_n^{\beta_n} \in \mathcal{B}$ ,

$$\begin{aligned} a^\alpha a^\beta &= \lambda_{\alpha,\beta} a^{\alpha+\beta} + f_{\alpha,\beta}, \\ \text{where } \lambda_{\alpha,\beta} &\in K^*, \quad a^{\alpha+\beta} = a_1^{\alpha_1+\beta_1} \cdots a_n^{\alpha_n+\beta_n}, \text{ and} \\ f_{\alpha,\beta} &\in K\text{-span}\mathcal{B} \text{ with } \mathbf{LM}(f_{\alpha,\beta}) \prec a^{\alpha+\beta} \text{ if } f_{\alpha,\beta} \neq 0, \end{aligned}$$

or alternatively, such that for all generators  $a_1, \dots, a_n$  of  $A$  and  $i < j$ ,

$$\begin{aligned} a_j a_i &= \lambda_{ji} a_i a_j + f_{ji}, \\ \text{where } \lambda_{ji} &\in K^*, \text{ and} \\ f_{ji} &\in K\text{-span}\mathcal{B} \text{ with } \mathbf{LM}(f_{ji}) \prec a_i a_j \text{ if } f_{ji} \neq 0, \end{aligned}$$

then  $A$  is said to be a *solvable polynomial algebra*.

**Remark** As one will see later, that the pure-ring-theoretical definition of a solvable polynomial algebra above may at least provide us with the following two advantages.

- (i) Solvable polynomial algebras can be characterized in a constructive way (see subsequent Section 1.4), so that more noncommutative algebras (and hence their modules) can be studied in a “solvable setting” (see [Wik1] for an introduction on “Decision problem” (or “Solvable problem”)).
- (ii) It is quite helpful in determining whether a solvable polynomial algebra  $A$  is an  $\mathbb{N}$ -graded algebra as specified in (Section 4.1 of Chapter 4), or an  $\mathbb{N}$ -filtered algebra as specified in (Section 5.1 of Chapter 5) respectively.

**Example (1)** By Definition 1.1.3, every commutative polynomial algebra  $K[x_1, \dots, x_n]$  in variables  $x_1, \dots, x_n$  over a field  $K$  is trivially a solvable polynomial algebra.

From [KR-W] and [Li1] we recall several most well-known noncommutative solvable polynomial algebras as follows.

**Example (2)** The  $n$ th Weyl algebra

The  $n$ th Weyl algebra  $A_n(k)$  over a field  $K$  is defined to be the  $K$ -algebra generated by  $2n$  generators  $x_1, \dots, x_n, y_1, \dots, y_n$  subject to the relations:

$$\begin{aligned} x_i x_j &= x_j x_i, \quad y_i y_j = y_j y_i, & 1 \leq i < j \leq n, \\ y_j x_i &= x_i y_j + \delta_{ij} \text{ the Kronecker delta,} & 1 \leq i, j \leq n. \end{aligned}$$

Historically, the Weyl algebra is the first “quantum algebra” ([Dir] 1926, [Wey] 1928). It is a well-known fact that if  $\text{char}K = 0$ , then

$A_n(k)$  coincides with the algebra of linear partial differential operators  $K[x_1, \dots, x_n, \partial_1, \dots, \partial_n]$  of the polynomial ring  $K[x_1, \dots, x_n]$  (or the ring of polynomial functions of the affine  $n$ -space  $\mathbf{A}_k^n$ ), where  $\partial_i = \frac{\partial}{\partial x_i}$ ,  $1 \leq i \leq n$ .

**Example (3)** The additive analogue of the Weyl algebra

This algebra was introduced in Quantum Physics in ([Kur] 1980) and studied in ([JBS] 1981), that is, the algebra  $A_n(q_1, \dots, q_n)$  generated over a field  $K$  by  $x_1, \dots, x_n, y_1, \dots, y_n$  subject to the relations:

$$\begin{aligned} x_i x_j &= x_j x_i, & y_i y_j &= y_j y_i, & 1 \leq i < j \leq n, \\ y_i x_i &= q_i x_i y_i + 1, & & & 1 \leq i \leq n, \\ x_j y_i &= y_i x_j, & & & i \neq j, \end{aligned}$$

where  $q_i \in K^*$ . If  $q_i = q \neq 0$ ,  $i = 1, \dots, n$ , then this algebra becomes the algebra of  $q$ -differential operators.

**Example (4)** The multiplicative analogue of the Weyl algebra

This is the algebra stemming from ([Jat] 1984 and [MP] 1988, where one may see why this algebra deserves its title), that is, the algebra  $\mathcal{O}_n(\lambda_{ji})$  generated over a field  $K$  by  $x_1, \dots, x_n$  subject to the relations:

$$x_j x_i = \lambda_{ji} x_i x_j, \quad 1 \leq i < j \leq n,$$

where  $\lambda_{ji} \in K^*$ . If  $n = 2$ , then  $\mathcal{O}_2(\lambda_{21})$  is the *quantum plane* in the sense of Manin [Man]. If  $\lambda_{ji} = q^{-2} \neq 0$  for some  $q \in K^*$  and all  $i < j$ , then  $\mathcal{O}_n(\lambda_{ji})$  becomes the coordinate ring of the so called *quantum affine  $n$ -space* (see [Sm]).

**Example (5)** The enveloping algebra of a finite dimensional Lie algebra

Let  $\mathfrak{g}$  be a finite dimensional vector space over the field  $k$  with basis  $\{x_1, \dots, x_n\}$  where  $n = \dim_k \mathfrak{g}$ . If there is a binary operation on  $\mathfrak{g}$ , called the *bracket product* and denoted  $[\ , \ ]$ , which is bilinear, i.e., for  $a, b, c \in \mathfrak{g}$ ,  $\lambda \in k$ ,

$$\begin{aligned} [a + b, c] &= [a, c] + [b, c] \\ [a, b + c] &= [a, b] + [a, c] \\ \lambda[a, b] &= [\lambda a, b] = [a, \lambda b], \end{aligned}$$

and satisfies:

$$\begin{aligned} [a, b] &= -[b, a], \quad a, b \in \mathfrak{g}, \\ [[a, b], c] + [[c, a], b] + [[b, c], a] &= 0 \quad \text{Jacobi identity, } a, b, c \in \mathfrak{g}, \end{aligned}$$

then  $\mathfrak{g}$  is called a finite dimensional Lie algebra over  $k$ . Note that  $[\ , \ ]$  need not satisfy the associative law. If  $[a, b] = [b, a]$  for every  $a, b$  in a Lie algebra  $\mathfrak{g}$ ,  $\mathfrak{g}$  is called *abelian*. The enveloping algebra of  $\mathfrak{g}$ , denoted  $U(\mathfrak{g})$ , is defined to be the *associative*  $k$ -algebra generated by  $x_1, \dots, x_n$  subject to the relations:

$$x_j x_i - x_i x_j = [x_j, x_i], \quad 1 \leq i < j \leq n.$$

For instance, the Heisenberg Lie algebra  $\mathfrak{h}$  has the  $k$ -basis  $\{x_i, y_j, z \mid i, j = 1, \dots, n\}$  and the bracket product is given by

$$\begin{aligned} [x_i, y_i] &= z, & 1 \leq i \leq n, \\ [x_i, x_j] &= [x_i, y_j] = [y_i, y_j] = 0, & i \neq j, \\ [z, x_i] &= [z, y_i] = 0, & 1 \leq i \leq n. \end{aligned}$$

**Example (6)** The  $q$ -Heisenberg algebra

This is the algebra stemming from ([Ber] 1992, [Ros] 1995) which has its root in  $q$ -calculus (e.g., [Wal] 1985), that is, the algebra  $\mathfrak{h}_n(q)$  generated over a field  $K$  by the set of elements  $\{x_i, y_i, z_i \mid i = 1, \dots, n\}$  subject to the relations:

$$\begin{aligned} x_i x_j &= x_j x_i, \quad y_i y_j = y_j y_i, \quad z_j z_i = z_i z_j, & 1 \leq i < j \leq n, \\ x_i z_i &= q z_i x_i, & 1 \leq i \leq n, \\ z_i y_i &= q y_i z_i, & 1 \leq i \leq n, \\ x_i y_i &= q^{-1} y_i x_i + z_i, & 1 \leq i \leq n, \\ x_i y_j &= y_j x_i, \quad x_i z_j = z_j x_i, \quad y_i z_j = z_j y_i, & i \neq j, \end{aligned}$$

where  $q \in K^*$ .

**Example (7)** The algebra of  $2 \times 2$  quantum matrices

This is the algebra  $M_q(2, K)$  introduced in ([Man] 1988), which is generated over a field  $K$  by  $a, b, c, d$  subject to the relations:

$$\begin{aligned} ba &= qab, \quad ca = qac, \quad dc = qcd, \\ db &= qbd, \quad cb = bc, \quad da - ad = (q - q^{-1})bc, \end{aligned}$$

where  $q \in K^*$ .

**Example (8)** The algebra  $\mathbf{U}$  in constructing Hayashi algebra

In order to get bosonic representations for the types of  $\mathbf{A}_n$  and  $\mathbf{C}_n$  of the Drinfeld-Jimbo quantum algebras, Hayashi introduced in ([Hay] 1990) the  $q$ -Weyl algebra  $\mathcal{A}_q^-$ , which is constructed as follows (see [Ber1]). Let

$\mathbf{U}$  be the algebra generated over the field  $K = \mathbb{C}$  by the set of elements  $\{x_i, y_i, z_i \mid i = 1, \dots, n\}$  subject to the relations:

$$\begin{aligned} x_j x_i &= x_i x_j, & y_j y_i &= y_i y_j, & z_j z_i &= z_i z_j, & 1 \leq i < j \leq n, \\ x_j y_i &= q^{-\delta_{ij}} y_i x_j, & z_j x_i &= q^{-\delta_{ij}} x_i z_j, & 1 \leq i, j \leq n, \\ z_j y_i &= y_i z_j, & i &\neq j, \\ z_i y_i - q^2 y_i z_i &= -q^2 x_i^2, & 1 \leq i \leq n, \end{aligned}$$

where  $q \in K^*$ . Then  $\mathcal{A}_q^- = S^{-1}\mathbf{U}$ ,

One may also use the technique given in the subsequent Section 1.4 to directly verify that the algebras listed above are solvable polynomial algebras.

### Basic properties

By Definition 1.1.1 and Definition 1.1.3, the properties listed in the next two propositions are straightforward.

**1.1.4. Proposition** Let  $A = K[a_1, \dots, a_n]$  be a solvable polynomial algebra with admissible system  $(\mathcal{B}, \prec)$ . The following statements hold.

(i) If  $f, g \in A$  with  $\mathbf{LM}(f) = a^\alpha$ ,  $\mathbf{LM}(g) = a^\beta$ , then

$$\mathbf{LM}(fg) = \mathbf{LM}(\mathbf{LM}(f)\mathbf{LM}(g)) = \mathbf{LM}(a^\alpha a^\beta) = a^{\alpha+\beta}.$$

(ii)  $A$  is a domain, that is,  $A$  has no (left and right) divisors of zero.

**Proof** Exercise. □

**1.1.5. Proposition** Let  $A_1 = K[a_1, \dots, a_n]$  and  $A_2 = K[b_1, \dots, b_m]$  be solvable polynomial algebras with admissible systems  $(\mathcal{B}_1, \prec_1)$  and  $(\mathcal{B}_2, \prec_2)$  respectively. Then  $A = A_1 \otimes_K A_2$  is a solvable polynomial algebra with the admissible system  $(\mathcal{B}, \prec)$ , where  $\mathcal{B} = \{a^\alpha \otimes b^\beta \mid a^\alpha \in \mathcal{B}_1, b^\beta \in \mathcal{B}_2\}$ , while  $\prec$  is defined on  $\mathcal{B}$  subject to the rule: for  $a^\alpha \otimes b^\beta, a^\gamma \otimes b^\eta \in \mathcal{B}$ ,

$$a^\alpha \otimes b^\beta \prec a^\gamma \otimes b^\eta \Leftrightarrow \begin{cases} a^\alpha \prec_1 a^\gamma; \\ \text{or} \\ a^\alpha = a^\gamma \text{ and } b^\beta \prec_2 b^\eta. \end{cases}$$

**Proof** Exercise.

## 1.2. Left Gröbner Bases of Left Ideals

Let  $A = K[a_1, \dots, a_n]$  be a solvable polynomial algebra with admissible system  $(\mathcal{B}, \prec)$ . In this section we introduce left Gröbner bases for left ideals of  $A$  via a left division algorithm in  $A$ , and we record some basic facts determined by left Gröbner bases. Moreover, minimal left Gröbner bases and reduced left Gröbner bases are discussed.

### Left division algorithm

Let  $a^\alpha, a^\beta \in \mathcal{B}$ . We say that  $a^\alpha$  *divides*  $a^\beta$  *from the left side*, denoted  $a^\alpha|_L a^\beta$ , if there exists  $a^\gamma \in \mathcal{B}$  such that

$$a^\beta = \mathbf{LM}(a^\gamma a^\alpha).$$

It follows from Proposition 1.1.4(i) that the division defined above is implementable.

Let  $F$  be a nonempty subset of  $A$ . Then the division of monomials defined above yields a subset of  $\mathcal{B}$ :

$$\mathcal{N}(F) = \{a^\alpha \in \mathcal{B} \mid \mathbf{LM}(f) \not|_L a^\alpha, f \in F\}.$$

If  $F = \{g\}$  consists of a single element  $g$ , then we simply write  $\mathcal{N}(g)$  in place of  $\mathcal{N}(F)$ . Also we write  $K\text{-span}\mathcal{N}(F)$  for the  $K$ -subspace of  $A$  spanned by  $\mathcal{N}(F)$ .

**1.2.1. Definition** Elements of  $\mathcal{N}(F)$  are referred to as *normal monomials* (mod  $F$ ). Elements of  $K\text{-span}\mathcal{N}(F)$  are referred to as *normal elements* (mod  $F$ ).

In view of Proposition 1.1.4(i), the left division we defined for monomials in  $\mathcal{B}$  can naturally be used to define a left division procedure for elements in  $A$ . More precisely, let  $f, g \in A$  with  $\mathbf{LC}(f) = \mu \neq 0$ ,  $\mathbf{LC}(g) = \lambda \neq 0$ . If  $\mathbf{LM}(g)|_L \mathbf{LM}(f)$ , i.e., there exists  $a^\alpha \in \mathcal{B}$  such that  $\mathbf{LM}(f) = \mathbf{LM}(a^\alpha \mathbf{LM}(g))$ , then put  $f_1 = f - \lambda^{-1} \mu a^\alpha g$ ; otherwise, put  $f_1 = f - \mathbf{LT}(f)$ . Note that in both cases we have  $f_1 = 0$ , or  $f_1 \neq 0$  and  $\mathbf{LM}(f_1) \prec \mathbf{LM}(f)$ . At this stage, let us refer to such a procedure of canceling the leading term of  $f$  as the *left division procedure by  $g$* . With  $f := f_1 \neq 0$ , we can repeat the left division procedure by  $g$  and so on.

This returns successively a descending sequence

$$\mathbf{LM}(f) \succ \mathbf{LM}(f_1) \succ \mathbf{LM}(f_2) \succ \dots$$

Since  $\prec$  is a well-ordering, it follows that such a division procedure terminates after a finite number of repetitions, and consequently  $f$  is expressed as

$$f = qg + r,$$

where  $q, r \in A$  with  $r \in K\text{-span}\mathcal{N}(g)$ , i.e.,  $r$  is normal (mod  $g$ ), such that either  $\mathbf{LM}(f) = \mathbf{LM}(qg)$  or  $\mathbf{LM}(f) = \mathbf{LM}(r)$ .

Furthermore, the left division procedure demonstrated above can be extended to a left division procedure by a finite subset  $G = \{g_1, \dots, g_s\}$  in  $A$ , which yields the following division algorithm:

---

**Algorithm-LDIV**

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INPUT:  $f, G = \{g_1, \dots, g_s\}$  with  $g_i \neq 0$  ( $1 \leq i \leq s$ )  
 OUTPUT:  $q_1, \dots, q_s, r$  such that  $r \in K\text{-span}\mathcal{N}(G)$ ,  $f = \sum_{i=1}^s q_i g_i + r$ ,  
 $\mathbf{LM}(q_i g_i) \preceq \mathbf{LM}(f)$  for  $q_i \neq 0$ ,  $\mathbf{LM}(r) \preceq \mathbf{LM}(f)$  if  $r \neq 0$   
 INITIALIZATION:  $q_1 := 0, q_2 := 0, \dots, q_s := 0; r := 0; h := f$   
 BEGIN  
   WHILE  $h \neq 0$  DO  
   IF there exist  $i$  and  $a^{\alpha(i)} \in \mathcal{B}$   
   such that  $\mathbf{LM}(h) = \mathbf{LM}(a^{\alpha(i)} \mathbf{LM}(g_i))$  THEN  
   choose  $i$  least such that  $\mathbf{LM}(h) = \mathbf{LM}(a^{\alpha(i)} \mathbf{LM}(g_i))$   
    $q_i := q_i + \mathbf{LC}(g_i)^{-1} \mathbf{LC}(h) a^{\alpha(i)}$   
    $h := h - \mathbf{LC}(g_i)^{-1} \mathbf{LC}(h) a^{\alpha(i)} g_i$   
   ELSE  
    $r := r + \mathbf{LT}(h)$   
    $h := h - \mathbf{LT}(h)$   
   END  
 END  
 END

---

**1.2.2. Definition** The element  $r$  obtained in **Algorithm-LDIV** is called a *remainder* of  $f$  on left division by  $G$ , and is denoted by  $\overline{f}^G$ , i.e.,  $\overline{f}^G = r$ . If  $\overline{f}^G = 0$ , then we say that  $f$  is *reduced to 0 (mod  $G$ )*.

**Remark** Note that the element  $r$  obtained in **Algorithm-LDIV** depends on how the order of elements in  $G$  is arranged. This implies that  $r$  may be different if a different order for elements of  $G$  is given (actually as in the commutative case, e.g. [AL2], P.31). That is why we use the phrase “a remainder” in the above definition.

Summing up, we have reached the following

**1.2.3. Theorem** Given a set of nonzero elements  $G = \{g_1, \dots, g_s\}$  and  $f$  in  $A$ , the **Algorithm-LDIV** produces elements  $q_1, \dots, q_s$ ,  $r \in A$  with  $r \in K\text{-span}\mathcal{N}(G)$ , such that  $f = \sum_{i=1}^s q_i g_i + r$  and  $\mathbf{LM}(q_i g_i) \preceq \mathbf{LM}(f)$  whenever  $q_i \neq 0$ ,  $\mathbf{LM}(r) \preceq \mathbf{LM}(f)$  if  $r \neq 0$ .  $\square$

### Left Gröbner bases

Theoretically the left division procedure by using a finite subset  $G$  of nonzero elements in  $A$  can be extended to a left division procedure by means of an *arbitrary proper subset*  $G$  of nonzero elements, for, it is a true statement that if  $f \in A$  with  $\mathbf{LM}(f) \neq 0$ , then either there exists  $g \in G$  such that  $\mathbf{LM}(g)|_L \mathbf{LM}(f)$  or such  $g$  does not exist. This leads to the following

**1.2.4. Proposition** Let  $N$  be a left ideal of  $A$  and  $G$  a proper subset of nonzero elements in  $N$ . The following two statements are equivalent:

- (i) If  $f \in N$  and  $f \neq 0$ , then there exists  $g \in G$  such that  $\mathbf{LM}(g)|_L \mathbf{LM}(f)$ .
- (ii) Every nonzero  $f \in N$  has a representation  $f = \sum_{j=1}^s q_j g_{i_j}$  with  $q_j \in A$  and  $g_{i_j} \in G$ , such that  $\mathbf{LM}(q_j g_{i_j}) \preceq \mathbf{LM}(f)$  whenever  $q_j \neq 0$ .  $\square$

**1.2.5. Definition** Let  $N$  be a left ideal of  $A$ . With respect to a given monomial ordering  $\prec$  on  $\mathcal{B}$ , a proper subset  $\mathcal{G}$  of nonzero elements in  $N$  is said to be a *left Gröbner basis* of  $N$  if  $\mathcal{G}$  satisfies one of the equivalent conditions of Proposition 1.2.4.

If  $\mathcal{G}$  is a left Gröbner basis of  $N$ , then the expression  $f = \sum_{j=1}^s q_j g_{i_j}$  appeared in Proposition 1.2.4(ii) is called a *left Gröbner representation* of  $f$ .

Clearly, if  $N$  is a left ideal of  $A$  and  $N$  has a left Gröbner basis  $\mathcal{G}$ , then  $\mathcal{G}$  is certainly a generating set of  $N$ , i.e.,  $N = \sum_{g \in \mathcal{G}} Ag$ . But the



converse is not necessarily true. For instance, in the solvable polynomial algebra  $A = \mathbb{C}[a_1, a_2, a_3]$  generated by  $\{a_1, a_2, a_3\}$  subject to the relations

$$a_2a_1 = 3a_1a_2, \quad a_3a_1 = a_1a_3, \quad a_3a_2 = 5a_2a_3,$$

let  $g_1 = a_1^2a_2 - a_3$ ,  $g_2 = a_2$ , and  $N = Ag_1 + Ag_2$ . Then since  $a_3 \in N$ ,  $G = \{g_1, g_2\}$  is not a left Gröbner basis with respect to any given monomial ordering  $\prec$  on  $\mathcal{B}$  such that  $\mathbf{LM}(g_1) = a_1^2a_2$ ,  $\mathbf{LM}(g_2) = a_2$ .

### The existence of left Gröbner bases

Noticing  $1 \in \mathcal{B}$ , if we define on  $\mathcal{B}$  the ordering:

$$a^\alpha \prec' a^\beta \Leftrightarrow a^\alpha|_{\mathbb{L}} a^\beta, \quad a^\alpha, a^\beta \text{ in } \mathcal{B},$$

then, by Definition 1.1.1, Proposition 1.1.4(i) and the divisibility defined for monomials, it is an easy exercise to check that  $\prec'$  is reflexive, anti-symmetric, transitive, and moreover,

$$a^\alpha \prec' a^\beta \text{ implies } a^\alpha \prec a^\beta.$$

Since the given monomial ordering  $\prec$  is a well-ordering on  $\mathcal{B}$ , it follows that every nonempty subset of  $\mathcal{B}$  has a minimal element with respect to the ordering  $\prec'$  on  $\mathcal{B}$ .

Now, let  $N \neq \{0\}$  be a left ideal of  $A$ ,  $\mathbf{LM}(N) = \{\mathbf{LM}(f) \mid f \in N\}$ , and  $\Omega = \{a^\alpha \mid a^\alpha \text{ is minimal in } \mathbf{LM}(N) \text{ w.r.t. } \prec'\}$ .

**1.2.6. Theorem** With the notation as above, the following statements hold.

- (i)  $\Omega \neq \emptyset$  and  $\Omega$  is a proper subset of  $\mathbf{LM}(N)$ .
- (ii) Let  $\mathcal{G} = \{g \in N \mid \mathbf{LM}(g) \in \Omega\}$ . Then  $\mathcal{G}$  is a left Gröbner basis of  $N$ .

**Proof** (i) That  $\Omega \neq \emptyset$  follows from  $N \neq \{0\}$  and the remark about  $\prec'$  we made above. Since  $N$  is a left ideal and  $A$  is a domain (Proposition 1.1.4), if  $f \in N$  and  $f \neq 0$ , then  $hf \in N$  for all nonzero  $h \in A$ . It follows from Proposition 1.1.4(i) that  $\mathbf{LM}(hf) \in \mathbf{LM}(N)$  and  $\mathbf{LM}(f)|_{\mathbb{L}} \mathbf{LM}(hf)$ , i.e.,  $\mathbf{LM}(f) \prec' \mathbf{LM}(hf)$  in  $\mathbf{LM}(N)$ . It is clear that if  $\mathbf{LM}(h) \neq 1$ , then  $\mathbf{LM}(hf) \neq \mathbf{LM}(f)$ . This shows that  $\Omega$  is a proper subset of  $\mathbf{LM}(N)$ .

(ii) By the definition of  $\prec'$  and the remark we made above the theorem, if  $f \in N$  with  $\mathbf{LM}(f) \neq 0$  and  $\mathbf{LM}(f) \notin \Omega$ , then there is some  $g \in \mathcal{G}$

such that  $\mathbf{LM}(g)|_L \mathbf{LM}(f)$ . Hence the selected  $\mathcal{G}$  is a left Gröbner basis for  $N$ .  $\square$

In Section 3 we will show that every nonzero left ideal  $N$  of  $A$  has a *finite* left Gröbner basis.

### Basic facts determined by left Gröbner bases

By referring to Proposition 1.1.4, Theorem 1.2.3 and Proposition 1.2.4, the foregoing discussion allows us to summarize some basic facts determined by left Gröbner bases, of which the detailed proof is left as an exercise.

**1.2.7. Proposition** Let  $N$  be a left ideal of  $A$ , and let  $\mathcal{G}$  be a left Gröbner basis of  $N$ . Then the following statements hold.

- (i) If  $f \in N$  and  $f \neq 0$ , then  $\mathbf{LM}(f) = \mathbf{LM}(qg)$  for some  $q \in A$  and  $g \in \mathcal{G}$ .
- (ii) If  $f \in A$  and  $f \neq 0$ , then  $f$  has a unique remainder  $\overline{f}^{\mathcal{G}}$  on division by  $\mathcal{G}$ .
- (iii) If  $f \in A$  and  $f \neq 0$ , then  $f \in N$  if and only if  $\overline{f}^{\mathcal{G}} = 0$ . Hence the membership problem for left ideals of  $A$  can be solved by using left Gröbner bases.
- (iv) As a  $K$ -vector space,  $A$  has the decomposition

$$A = N \oplus K\text{-span}\mathcal{N}(\mathcal{G}).$$

- (v) As a  $K$ -vector space,  $A/N$  has the  $K$ -basis

$$\overline{\mathcal{N}(\mathcal{G})} = \{\overline{a^\alpha} \mid a^\alpha \in \mathcal{N}(\mathcal{G})\},$$

where  $\overline{a^\alpha}$  denotes the coset represented by  $a^\alpha$  in  $A/N$ .

- (vi)  $\mathcal{N}(\mathcal{G})$  is a finite set, or equivalently,  $\dim_K A/N < \infty$ , if and only if for each  $i = 1, \dots, n$ , there exists  $g_{j_i} \in \mathcal{G}$  such that  $\mathbf{LM}(g_{j_i}) = a_i^{m_i}$ , where  $m_i \in \mathbb{N}$ .  $\square$

### Minimal and reduced left Gröbner bases

**1.2.8. Definition** Let  $N \neq \{0\}$  be a left ideal of  $A$  and let  $\mathcal{G}$  be a left gröbner basis of  $N$ . If any proper subset of  $\mathcal{G}$  cannot be a left Gröbner basis of  $N$ , then  $\mathcal{G}$  is called a *minimal left Gröbner basis* of  $N$ .

**1.2.9. Proposition** Let  $N \neq \{0\}$  be a left ideal of  $A$ . A left Gröbner basis  $\mathcal{G}$  of  $N$  is minimal if and only if  $\mathbf{LM}(g_i) \not\ll_{\mathbf{L}} \mathbf{LM}(g_j)$  for all  $g_i, g_j \in \mathcal{G}$  with  $g_i \neq g_j$ .

**Proof** Suppose that  $\mathcal{G}$  is minimal. If there were  $g_i \neq g_j$  in  $\mathcal{G}$  such that  $\mathbf{LM}(g_i) \ll_{\mathbf{L}} \mathbf{LM}(g_j)$ , then since the left division is transitive, the proper subset  $\mathcal{G}' = \mathcal{G} - \{g_j\}$  of  $\mathcal{G}$  would form a left Gröbner basis of  $N$ . This contradicts the minimality of  $\mathcal{G}$ .

Conversely, if the condition  $\mathbf{LM}(g_i) \not\ll_{\mathbf{L}} \mathbf{LM}(g_j)$  holds for all  $g_i, g_j \in \mathcal{G}$  with  $g_i \neq g_j$ , then the definition of a left Gröbner basis entails that any proper subset of  $\mathcal{G}$  cannot be a left Gröbner basis of  $N$ . This shows that  $\mathcal{G}$  is minimal.  $\square$

**1.2.10. Corollary** Let  $N \neq \{0\}$  be a left ideal of  $A$ , and let the notation be as in Theorem 1.2.6.

- (i) The left Gröbner basis  $\mathcal{G}$  obtained there is indeed a minimal left Gröbner basis for  $N$ .
- (ii) If  $G$  is any left Gröbner basis of  $N$ , then

$$\Omega = \mathbf{LM}(\mathcal{G}) = \{\mathbf{LM}(g) \mid g \in \mathcal{G}\} \subseteq \mathbf{LM}(G) = \{\mathbf{LM}(g') \mid g' \in G\}.$$

Therefore, any two minimal left Gröbner bases of  $N$  have the same set of leading monomials  $\Omega$ .

**Proof** This follows immediately from the definition of a left Gröbner basis, the construction of  $\Omega$  and Proposition 1.2.9.  $\square$

We next introduce the notion of a reduced left Gröbner basis.

**1.2.11. Definition** Let  $N \neq \{0\}$  be a left ideal of  $A$  and let  $\mathcal{G}$  be a left gröbner basis of  $N$ . If  $\mathcal{G}$  satisfies the following conditions:

- (1)  $\mathcal{G}$  is a minimal left Gröbner basis;
  - (2)  $\mathbf{LC}(g) = 1$  for all  $g \in \mathcal{G}$ ;
  - (3) For every  $g \in \mathcal{G}$ ,  $h = g - \mathbf{LM}(g)$  is a normal element (mod  $\mathcal{G}$ ), i.e.,  $h \in K\text{-span}\mathcal{N}(\mathcal{G})$ ,
- then  $\mathcal{G}$  is called a *reduced left Gröbner basis* of  $N$ .

**1.2.12. Proposition** Every nonzero left ideal  $N$  of  $A$  has a unique reduced left Gröbner basis. Therefore, two left ideals  $N_1, N_2$  have the

same reduced left Gröbner basis if and only if  $N_1 = N_2$ .

**Proof** Note that if  $\mathcal{G}$  and  $\mathcal{G}'$  are reduced left Gröbner bases for  $I$ , then  $\mathbf{LM}(\mathcal{G}) = \mathbf{LM}(\mathcal{G}')$  by Corollary 1.2.10. If  $\mathbf{LM}(g_i) = \mathbf{LM}(g'_j)$  then  $g_i - g'_j \in N \cap K\text{-span}\mathcal{N}(\mathcal{G}) = N \cap K\text{-span}\mathcal{N}(\mathcal{G}')$ . It follows from Proposition 1.2.7(iv) that  $g_i = g'_j$ . Hence  $\mathcal{G} = \mathcal{G}'$ . By the uniqueness, the second assertion is clear.  $\square$

Since we will see in Section 3 that every left ideal  $N$  of  $A$  has a finite left Gröbner basis, a minimal left Gröbner basis, thereby the reduced left Gröbner basis for  $N$ , can be obtained in an algorithmic way. More precisely, the next proposition holds true.

**1.2.13. Proposition** Let  $N \neq \{0\}$  be a left ideal of  $A$ , and let  $\mathcal{G} = \{g_1, \dots, g_m\}$  be a finite left Gröbner basis of  $N$ .

(i) The subset  $\mathcal{G}_0 = \{g_i \in \mathcal{G} \mid \mathbf{LM}(g_i) \text{ is minimal in } \mathbf{LM}(\mathcal{G}) \text{ w.r.t. } \prec'\}$  of  $\mathcal{G}$  forms a minimal left Gröbner basis of  $N$  (see the definition of  $\prec'$  given before Theorem 1.2.6). An algorithm written in pseudo-code is omitted here.

(ii) With  $\mathcal{G}_0$  as in (i) above, we may assume, without loss of generality, that  $\mathcal{G}_0 = \{g_1, \dots, g_s\}$  with  $\mathbf{LC}(g_i) = 1$  for  $1 \leq i \leq s$ . Put  $\mathcal{G}_1 = \{g_2, \dots, g_s\}$  and  $h_1 = \overline{g_1}^{\mathcal{G}_1}$ . Then  $\mathbf{LM}(h_1) = \mathbf{LM}(g_1)$ . Put  $\mathcal{G}_2 = \{h_1, g_3, \dots, g_s\}$  and  $h_2 = \overline{g_2}^{\mathcal{G}_2}$ . Then  $\mathbf{LM}(h_2) = \mathbf{LM}(g_2)$ . Put  $\mathcal{G}_3 = \{h_1, h_2, g_4, \dots, g_s\}$  and  $h_3 = \overline{g_3}^{\mathcal{G}_3}$ , and so on. The last obtained  $\mathcal{G}_{s+1} = \{h_1, \dots, h_{s-1}\} \cup \{h_s\}$  is then the reduced left Gröbner basis. An algorithm written in pseudo-code is omitted here.

**Proof** This can be verified directly, so we leave it as an exercise.  $\square$

### 1.3. The Noetherianess

Let  $A = K[a_1, \dots, a_n]$  be a solvable polynomial algebra with admissible system  $(\mathcal{B}, \prec)$ . In this section we show that  $A$  is a (left and right) Noetherian  $K$ -algebra by showing that every (left, right) ideal of  $A$  has a finite (left, right) Gröbner basis.

The following lemma, which will be essential not only in proving our next theorem but also in proving the existence of finite left Gröbner bases and the termination property of Buchberger's algorithm

for modules over solvable polynomial algebras (Section 3 of Chapter 2), is usually attributed to the American algebraist L. E. Dickson ([http://en.wikipedia.org/wiki/Dickson's\\_lemma](http://en.wikipedia.org/wiki/Dickson's_lemma)). For a detailed argument on Dickson's lemma in commutative computational algebra, we refer to ([BW], 4.4).

**1.3.1. Lemma** (Dickson's Lemma) Every subset  $S$  of  $\mathbb{N}^n$  has a finite subset  $B$  such that for each  $(\alpha_1, \dots, \alpha_n) \in S$ , there exists  $(\gamma_1, \dots, \gamma_n) \in B$  with  $\gamma_i \leq \alpha_i$  for  $1 \leq i \leq n$ .  $\square$

Let  $a^\alpha, a^\beta \in \mathcal{B}$  with  $\alpha = (\alpha_1, \dots, \alpha_n)$ ,  $\beta = (\beta_1, \dots, \beta_n) \in \mathbb{N}^n$ . Recall that  $a^\alpha \prec' a^\beta$  if and only if  $a^\alpha|_L a^\beta$ , while the latter is defined subject to the property that  $a^\beta = \mathbf{LM}(a^\gamma a^\alpha)$  for some  $a^\gamma \in \mathcal{B}$ . Thus, by Proposition 1.1.4(i),  $a^\alpha \prec' a^\beta$  is actually equivalent to the property that  $\alpha_i \leq \beta_i$  for  $1 \leq i \leq n$ . Also note that the correspondence  $\alpha = (\alpha_1, \dots, \alpha_n) \longleftrightarrow a^\alpha$  gives the bijection between  $\mathbb{N}^n$  and  $\mathcal{B}$ . Combining this observation with Dickson's lemma, the following result is obtained.

**1.3.2. Theorem** With notations as in Theorem 1.2.6, let  $N \neq \{0\}$  be a left ideal of  $A$ ,  $\mathbf{LM}(N) = \{\mathbf{LM}(f) \mid f \in N\}$ ,  $\Omega = \{a^\alpha \in \mathbf{LM}(N) \mid a^\alpha \text{ is minimal w.r.t. } \prec'\}$ , and  $\mathcal{G} = \{g \in N \mid \mathbf{LM}(g) \in \Omega\}$ . Then  $\Omega$  is a finite subset of  $\mathbf{LM}(N)$ , thereby  $\mathcal{G}$  is a finite left Gröbner basis of  $N$ .  $\square$

Since every solvable polynomial algebra has a (two-sided) monomial ordering, a right division algorithm and a theory of right Gröbner bases for right ideals hold true as well, we then have the following

**1.3.3. Corollary** Every solvable polynomial algebra  $A$  is (left and right) Noetherian.  $\square$

In the next chapter we will see that the Buchberger algorithm, which computes a finite Gröbner basis for a finitely generated commutative polynomial ideal, has a complete noncommutative version (**Algorithm-LGB** presented in Section 3 of Chapter 2), that computes a finite left Gröbner basis for a finitely generated submodule of a free module over a solvable polynomial algebra.

## 1.4. A Constructive Characterization

From Definition 1.1.3 we see that the two conditions (S1) and (S2), which determine a solvable polynomial algebra  $A = K[a_1, \dots, a_n]$ , are mutually *independent* factors. In this section we give a characterization of solvable polynomial algebras by employing Gröbner bases of ideals in free algebras, so that solvable polynomial algebras are completely recognizable and constructible in a computational way.

To make the text self-contained, we start by recalling some basics on Gröbner bases of *two-sided ideals* in a free  $K$ -algebra  $K\langle X \rangle = K\langle X_1, \dots, X_n \rangle$  on  $X = \{X_1, \dots, X_n\}$ .

### Division algorithm in $K\langle X \rangle$

Let  $\mathbb{B} = \{1, X_{i_1} \cdots X_{i_s} \mid X_{i_j} \in X, s \geq 1\}$  be the standard  $K$ -basis of  $K\langle X \rangle$ . For convenience, elements of  $\mathbb{B}$  are also referred to as *monomials*, and we use capital letters  $U, V, W, S, \dots$  to denote monomials in  $\mathbb{B}$ . Recall that a *monomial ordering*  $\prec_X$  on  $\mathbb{B}$  (or equivalently on  $K\langle X \rangle$ ) is a well-ordering such that for any  $W, U, V, S \in \mathbb{B}$ ,

- (1)  $U \prec_X V$  implies  $WU \prec_X WV$ ,  $US \prec_X VS$ , or equivalently,  $WUS \prec_X WVS$ ;
- (2) if  $U \neq V$ , then  $V = WUS$  implies  $U \prec_X V$  (thereby  $1 \prec_X W$  for all  $1 \neq W \in \mathbb{B}$ ).

If  $\prec_X$  is a monomial ordering on  $\mathbb{B}$ , then the data  $(\mathbb{B}, \prec_X)$  is referred to as an *admissible system* of  $K\langle X \rangle$ .

Note that for any given  $n$ -tuple  $(m_1, \dots, m_n) \in \mathbb{N}^n$ , a *weighted degree function*  $d(\cdot)$  is well defined on nonzero elements of  $K\langle X \rangle$ , namely, by assigning each  $X_i$  the degree  $d(X_i) = m_i$ ,  $1 \leq i \leq n$ , we may define for each  $W = X_{i_1} \cdots X_{i_s} \in \mathbb{B}$  the degree  $d(W) = m_{i_1} + \cdots + m_{i_s}$ , and for each nonzero  $f = \sum_{i=1}^s \lambda_i W_i \in K\langle X \rangle$  with  $\lambda_i \in K^*$  and  $W_i \in \mathbb{B}$ , the degree of  $f$  is then defined as

$$d(f) = \max\{d(W_i) \mid 1 \leq i \leq s\}.$$

If  $d(X_i) = m_i > 0$  for  $1 \leq i \leq n$ , then  $d(\cdot)$  is referred to as a *positive-degree function* on  $K\langle X \rangle$ .

Let  $d(\cdot)$  be a positive-degree function on  $K\langle X \rangle$ . If  $\prec_X$  is a monomial

ordering on  $\mathbb{B}$  such that for all  $U, V \in \mathbb{B}$ ,

$$U \prec_X V \text{ implies } d(U) \leq d(V),$$

then, as in Section 1, we call  $\prec_X$  a *graded monomial ordering* with respect to  $d(\cdot)$ . For instance, with respect to any given positive-degree function  $d(\cdot)$ , the lexicographic graded ordering  $\prec_{grlex}$  on  $\mathbb{B}$  can be defined as follows. For  $U, V \in \mathbb{B}$ ,

$$U \prec_{grlex} V \Leftrightarrow \begin{cases} d(U) < d(V); \\ \text{or} \\ d(U) = d(V) \text{ and } U \prec_{lex} V, \end{cases}$$

where the lexicographic ordering  $\prec_{lex}$  on  $\mathbb{B}$  may be defined by ordering  $X_1, \dots, X_n$  arbitrarily, say  $X_{i_1} \prec_{lex} X_{i_2} \prec_{lex} \dots \prec_{lex} X_{i_n}$ , i.e., if  $U = X_{\ell_1} \dots X_{\ell_s}$ ,  $V = X_{t_1} \dots X_{t_m} \in \mathbb{B}$  and  $s \leq m$ , then

$$U \prec_{lex} V \Leftrightarrow \begin{cases} s < m, X_{\ell_k} = X_{t_k}, 1 \leq k \leq s; \\ \text{or} \\ s = m, \text{ there exists } p \leq s \text{ such that } X_{\ell_k} = X_{t_k} \text{ for } k < p \\ \text{and } X_{\ell_p} \prec_{lex} X_{t_p}. \end{cases}$$

At this point we should point out that though the ordering  $\prec_{lex}$  is a total ordering on  $\mathbb{B}$ , *it is not a monomial ordering* on  $\mathbb{B}$ . For instance, considering the free algebra  $K\langle X_1, X_2 \rangle$  with  $X_1 \prec_{lex} X_2$ , we have

$$X_2 \succ_{lex} X_1 X_2 \succ_{lex} X_1 X_1 X_2 \succ_{lex} X_1 X_1 X_1 X_2 \succ_{lex} \dots$$

This shows that  $\prec_{lex}$  is not a well-ordering on  $K\langle X \rangle$ .

One may refer to loc. cit. (e.g. [Gr]) for more monomial orderings on  $K\langle X \rangle$ .

Let  $\prec_X$  be a monomial ordering on  $\mathbb{B}$ . If  $f \in K\langle X \rangle$  is such that  $f = \sum_{i=1}^m \lambda_i W_i$  with  $\lambda_i \in K^*$ ,  $W_i \in \mathbb{B}$ , and  $W_1 \prec_X W_2 \prec_X \dots \prec_X W_m$ , then we write  $\mathbf{LM}(f) = W_m$  for the *leading monomial* of  $f$ ,  $\mathbf{LC}(f) = \lambda_m$  for the *leading coefficient* of  $f$ , and we write  $\mathbf{LT}(f) = \lambda W_m$  for the *leading term* of  $f$ .

Note that  $\mathbb{B}$  forms a multiplicative monoid with the identity 1 and the multiplication in  $\mathcal{B}$  is just simply the concatenation of monomials, i.e., if  $U, V \in \mathbb{B}$  with  $U = X_{i_1} \dots X_{i_p}$ ,  $V = X_{j_1} \dots X_{j_q}$ , then  $UV =$

$X_{i_1} \cdots X_{i_p} X_{j_1} \cdots X_{j_q} \in \mathbb{B}$ . Thereby if there exist  $W, S \in \mathbb{B}$  such that  $V = WUS$ , then we say that  $U$  divides  $V$ , denoted  $U|V$ .

Let  $F$  be a nonempty subset of  $K\langle X \rangle$ . Then the division of monomials defined above yields a subset of  $\mathbb{B}$ :

$$\mathcal{N}(F) = \{W \in \mathbb{B} \mid \mathbf{LM}(f) \nmid W, f \in F\}.$$

If  $F = \{g\}$  consists of a single element  $g$ , then we simply write  $\mathcal{N}(g)$  in place of  $\mathcal{N}(F)$ . Also we write  $K\text{-span}\mathcal{N}(F)$  for the  $K$ -subspace of  $K\langle X \rangle$  spanned by  $\mathcal{N}(F)$ .

**1.4.1. Definition** Elements of  $\mathcal{N}(F)$  are referred to as *normal monomials* (mod  $F$ ). Elements of  $K\text{-span}\mathcal{N}(F)$  are referred to as *normal elements* (mod  $F$ ).

Given a monomial ordering  $\prec_x$  on  $\mathbb{B}$ , if  $f \in K\langle X \rangle$  and  $U, V \in \mathbb{B}$ , then the definition of  $\prec_x$  entails that  $\mathbf{LM}(UfV) = U\mathbf{LM}(f)V$ . So, the division we defined for monomials in  $\mathbb{B}$  can naturally be used to define a division procedure for elements in  $K\langle X \rangle$ . More precisely, let  $f, g \in K\langle X \rangle$  with  $\mathbf{LC}(f) = \mu \neq 0$ ,  $\mathbf{LC}(g) = \lambda \neq 0$ . If  $\mathbf{LM}(g) \mid \mathbf{LM}(f)$ , i.e., there exists  $U, V \in \mathbb{B}$  such that  $\mathbf{LM}(f) = U\mathbf{LM}(g)V$ , then put  $f_1 = f - \lambda^{-1}\mu UgV$ ; otherwise, put  $f_1 = f - \mathbf{LT}(f)$ . Note that in both cases we have  $f_1 = 0$ , or  $f_1 \neq 0$  and  $\mathbf{LM}(f_1) \prec_x \mathbf{LM}(f)$ . At this stage, let us refer to such a procedure of canceling the leading term of  $f$  as the *division procedure by  $g$* . With  $f := f_1 \neq 0$ , we can repeat the division procedure by  $g$  and so on. This, in turn, gives rise to

$$\mathbf{LM}(f_{i+1}) \prec_x \mathbf{LM}(f_i) \prec_x \cdots \prec_x \mathbf{LM}(f_1) \prec_x \mathbf{LM}(f), \quad i \geq 2.$$

Since  $\prec_x$  is a well-ordering, it follows that such a division procedure terminates after a finite number of repetitions, and consequently  $f$  is expressed as

$$f = \sum_i^t \lambda_i U_i g V_i + r,$$

where  $\lambda_i \in K^*$ ,  $U_i, V_i \in \mathbb{B}$ , and  $r \in K\text{-span}\mathcal{N}(g)$ , i.e.,  $r$  is normal (mod  $g$ ), such that either  $\mathbf{LM}(f) = \max\{\mathbf{LM}(U_i g V_i) \mid 1 \leq i \leq t\}$  or  $\mathbf{LM}(f) = \mathbf{LM}(r)$ .



Furthermore, the division procedure demonstrated above can be extended to a division procedure canceling the leading term of  $f$  by a finite subset  $G = \{g_1, \dots, g_s\}$ , which therefore gives rise to an effective division algorithm in  $K\langle X \rangle$ , that is, we have reached the following

**1.4.2. Theorem** Given a set of nonzero elements  $G = \{g_1, \dots, g_s\}$  and  $f$  in  $A$ , the **Algorithm-DIV** given below produces finitely many  $\lambda_{ij} \in K^*$ ,  $U_{ij}, V_{ij} \in \mathbb{B}$ , and an  $r \in K\text{-span}\mathcal{N}(G)$ , such that  $f = \sum_{i,j} \lambda_{ij} U_{ij} g_j V_{ij} + r$  and  $\mathbf{LM}(U_{ij} g_j V_{ij}) \preceq \mathbf{LM}(f)$ ,  $\mathbf{LM}(r) \preceq \mathbf{LM}(f)$  if  $r \neq 0$ .

---

**Algorithm-DIV**

---

INPUT:  $f$ ,  $G = \{g_1, \dots, g_s\}$  with  $g_i \neq 0$  ( $1 \leq i \leq s$ )

OUTPUT:  $\lambda_{ij} \in K^*$ ,  $U_{ij}, V_{ij} \in \mathbb{B}$ ,  $r \in K\text{-span}\mathcal{N}(G)$

INITIALIZATION:  $i := 0$ ;  $r := 0$ ;  $h := f$

$\Lambda_1 := \emptyset, \dots, \Lambda_s := \emptyset$ ;  $Q_1 := \emptyset, \dots, Q_s := \emptyset$

```

BEGIN
  WHILE  $h \neq 0$  DO
    IF there exist  $j$  and  $U, V \in \mathbb{B}$ 
      such that  $\mathbf{LM}(h) = U\mathbf{LM}(g_j)V$  THEN
      choose  $j$  least such that  $\mathbf{LM}(h) = U\mathbf{LM}(g_j)V$ 
       $i := i + 1$ ,  $\lambda_{ij} := \mathbf{LC}(g_j)^{-1}\mathbf{LC}(h)$ ,  $U_{ij} := U$ ,  $V_{ij} := V$ 
       $\Lambda_j := \Lambda_j \cup \{\lambda_{ij}\}$ ,  $Q_j := Q_j \cup \{U_{ij}, V_{ij}\}$ 
       $h := h - \mathbf{LC}(g_j)^{-1}\mathbf{LC}(h)U_{ij}g_jV_{ij}$ 
    ELSE
       $r := r + \mathbf{LT}(h)$ 
       $h := h - \mathbf{LT}(h)$ 
    END
  END
END

```

---

**1.4.3. Definition** The element  $r$  obtained in **Algorithm-DIV** is called a *remainder* of  $f$  on division by  $G$ , and is denoted by  $\overline{f}^G$ , i.e.,  $\overline{f}^G = r$ . If  $\overline{f}^G = 0$ , then we say that  $f$  is *reduced to 0 (mod  $G$ )*.

**Remark** Actually as in the case of (Section 2, **Algorithm-LDIV**), the element  $r$  obtained in **Algorithm-DIV** depends on how the order of elements in  $G$  is arranged. This implies that the  $r$  may not be unique. That is why the phrase “a remainder” is used in the above definition.

### Gröbner bases for two-sided ideals of $K\langle X \rangle$

For the remainder of this section, ideals of  $K\langle X \rangle$  are always meant two-sided ideals; if  $M \subset K\langle X \rangle$  is a nonempty subset, then we write  $\mathbf{LM}(M) = \{\mathbf{LM}(f) \mid f \in M\}$  for the set of leading monomials of  $M$ , and we write  $I = \langle M \rangle$  for the two-sided ideal  $I$  of  $K\langle X \rangle$  generated by  $M$ ; moreover, we fix an admissible system  $(\mathbb{B}, \prec_x)$  of  $K\langle X \rangle$ .

As with a solvable polynomial algebra in Section 2, in principle the division procedure by using a finite subset  $G$  of nonzero elements in  $K\langle X \rangle$  can be extended to a left division procedure by means of an *arbitrary proper subset*  $G$  of nonzero elements, for, it is a true statement that if  $f \in K\langle X \rangle$  with  $\mathbf{LM}(f) \neq 0$ , then either there exists  $g \in G$  such that  $\mathbf{LM}(g) \mid \mathbf{LM}(f)$  or such  $g$  does not exist. This leads to the following

**1.4.4. Proposition** Let  $I$  be an ideal of  $K\langle X \rangle$  and  $G$  a proper subset of nonzero elements in  $I$ . The following statements are equivalent:

- (i) If  $f \in I$  and  $f \neq 0$ , then there exists  $g \in G$  such that  $\mathbf{LM}(g) | \mathbf{LM}(f)$ .
- (ii) Every nonzero  $f \in I$  has a finite representation  $f = \sum_{i,j} \lambda_{ij} U_{ij} g_j V_{ij}$  with  $\lambda_{ij} \in K^*$ ,  $U_{ij}, V_{ij} \in \mathbb{B}$  and  $g_j \in G$ , such that  $\mathbf{LM}(U_{ij} g_j V_{ij}) \preceq_x \mathbf{LM}(f)$ .
- (iii)  $\langle \mathbf{LM}(I) \rangle = \langle \mathbf{LM}(G) \rangle$ . □

**1.4.5. Definition** Let  $I$  be an ideal of  $K\langle X \rangle$ . With respect to a given monomial ordering  $\prec_x$  on  $\mathbb{B}$ , a proper subset  $\mathcal{G}$  of nonzero elements in  $I$  is said to be a *Gröbner basis* of  $I$  if  $\mathcal{G}$  satisfies one of the equivalent conditions of Proposition 1.4.4.

If  $\mathcal{G}$  is a Gröbner basis of  $I$ , then the expression  $f = \sum_{i,j} \lambda_{ij} U_{ij} g_j V_{ij}$  appeared in Proposition 1.4.4(ii) is called a *Gröbner representation* of  $f$ .

Clearly, if  $I$  is an ideal of  $K\langle X \rangle$  and  $I$  has a Gröbner basis  $\mathcal{G}$ , then  $\mathcal{G}$  is certainly a generating set of  $I$ , i.e.,  $I = \langle \mathcal{G} \rangle$ . But the converse is not necessarily true. For instance, let  $g_1 = X_1^2 X_2 - X_3$ ,  $g_2 = X_2$ , and  $I = \langle g_1, g_2 \rangle$ . Then since  $X_3 \in I$ ,  $G = \{g_1, g_2\}$  is not a Gröbner basis with respect to any given monomial ordering  $\prec_x$  on  $\mathbb{B}$  such that  $\mathbf{LM}(g_1) = X_1^2 X_2$ ,  $\mathbf{LM}(g_2) = X_2$ .

### The existence of Gröbner bases

Noticing  $1 \in \mathbb{B}$ , if we define on  $\mathbb{B}$  the ordering:

$$U \prec V \Leftrightarrow U|V, \quad U, V \text{ in } \mathbb{B},$$

then, by the divisibility defined for monomials, it is easy to see that  $\prec$  is reflexive, anti-symmetric, transitive, and moreover,

$$U \prec V \text{ implies } U \prec_x V.$$

Since the given monomial ordering  $\prec_x$  is a well-ordering on  $\mathbb{B}$ , it follows that every nonempty subset of  $\mathbb{B}$  has a minimal element with respect to the ordering  $\prec$  on  $\mathbb{B}$ .

Now, let  $I$  be a nonzero ideal of  $K\langle X \rangle$ ,  $\mathbf{LM}(I) = \{\mathbf{LM}(f) \mid f \in I\}$ , and  $\Omega = \{U \in \mathbf{LM}(I) \mid U \text{ is minimal w.r.t. } \prec\}$ .

**1.4.6. Theorem** With the notation as above, the following statements hold.

- (i)  $\Omega \neq \emptyset$  and  $\Omega$  is a proper subset of  $\mathbf{LM}(I)$ .
- (ii) Let  $\mathcal{G} = \{g \in I \mid \mathbf{LM}(g) \in \Omega\}$ . Then  $\mathcal{G}$  is a Gröbner basis of  $I$ .

**Proof** (i) That  $\Omega \neq \emptyset$  follows from  $I \neq \{0\}$  and the remark about  $\prec$  we made above. Since  $I$  is a two-sided ideal and  $K\langle X \rangle$  is a domain, if  $f \in I$  then  $hfg \in I$ , thereby  $\mathbf{LM}(hfg) \in \mathbf{LM}(I)$  for all  $h, g \in K\langle X \rangle$ . Also note that the leading monomials of elements in  $K\langle X \rangle$  are taken with respect to the given monomial ordering  $\prec_x$ . It follows that  $\mathbf{LM}(hfg) = \mathbf{LM}(h)\mathbf{LM}(f)\mathbf{LM}(g)$  which implies  $\mathbf{LM}(f) \mid \mathbf{LM}(hfg)$ , thereby  $\mathbf{LM}(f) \prec \mathbf{LM}(hfg)$  in  $\mathbf{LM}(I)$ . It is clear that if  $\mathbf{LM}(h) \neq 1$  or  $\mathbf{LM}(g) \neq 1$ , then  $\mathbf{LM}(hfg) \neq \mathbf{LM}(f)$ . This shows that  $\Omega$  is a proper subset of  $\mathbf{LM}(I)$ .

(ii) By the definition of  $\prec$  and the remark we made above the theorem, if  $f \in I$  with  $\mathbf{LM}(f) \neq 0$  and  $\mathbf{LM}(f) \notin \Omega$ , then there is some  $g \in \mathcal{G}$  such that  $\mathbf{LM}(g) \mid \mathbf{LM}(f)$ . Hence the selected  $\mathcal{G}$  is a Gröbner basis for  $I$ .  $\square$

### Basic facts determined by Gröbner bases

By referring to Proposition 1.4.4, Definition 1.4.5 and the foregoing discussion, the next proposition summarizes some basic facts determined by Gröbner bases of ideals in  $K\langle X \rangle$ , of which the detailed proof is left as an exercise.

**1.4.7. Proposition** Let  $I$  be an ideal of  $K\langle X \rangle$ , and let  $\mathcal{G}$  be a Gröbner basis of  $I$ . Then the following statements hold.

- (i) If  $f \in I$  and  $f \neq 0$ , then  $\mathbf{LM}(f) = \mathbf{LM}(UgV)$  for some  $U, V \in \mathbb{B}$  and  $g \in \mathcal{G}$ .
- (ii) If  $f \in K\langle X \rangle$  and  $f \neq 0$ , then  $f$  has a unique remainder  $\overline{f}^{\mathcal{G}}$  on division by  $\mathcal{G}$ .
- (iii) If  $f \in K\langle X \rangle$  and  $f \neq 0$ , then  $f \in I$  if and only if  $\overline{f}^{\mathcal{G}} = 0$ . Hence the membership problem for ideals of  $K\langle X \rangle$  can be solved by using Gröbner bases.
- (iv) As a  $K$ -vector space,  $K\langle X \rangle$  has the decomposition

$$K\langle X \rangle = I \oplus K\text{-span}\mathcal{N}(\mathcal{G}).$$

- (v) As a  $K$ -vector space,  $K\langle X \rangle/I$  has the  $K$ -basis

$$\overline{\mathcal{N}(\mathcal{G})} = \{\overline{U} \mid U \in \mathcal{N}(\mathcal{G})\},$$

where  $\overline{U}$  denotes the coset represented by  $U$  in  $K\langle X\rangle/I$ .  $\square$

### Minimal and reduced Gröbner bases

**1.4.8. Definition** Let  $I \neq \{0\}$  be an ideal of  $K\langle X\rangle$  and let  $\mathcal{G}$  be a gröbner basis of  $I$ . If any proper subset of  $\mathcal{G}$  cannot be a Gröbner basis of  $I$ , then  $\mathcal{G}$  is called a *minimal Gröbner basis* of  $I$ .

**1.4.9. Proposition** Let  $I \neq \{0\}$  be an ideal of  $K\langle X\rangle$ . A Gröbner basis  $\mathcal{G}$  of  $I$  is minimal if and only if  $\mathbf{LM}(g_i) \nmid \mathbf{LM}(g_j)$  for all  $g_i, g_j \in \mathcal{G}$  with  $g_i \neq g_j$ .

**Proof** Suppose that  $\mathcal{G}$  is minimal. If there were  $g_i \neq g_j$  in  $\mathcal{G}$  such that  $\mathbf{LM}(g_i) \mid \mathbf{LM}(g_j)$ , then since the division of monomials is transitive, the proper subset  $\mathcal{G}' = \mathcal{G} - \{g_j\}$  of  $\mathcal{G}$  would form a Gröbner basis of  $I$ . This contradicts the minimality of  $\mathcal{G}$ .

Conversely, if the condition  $\mathbf{LM}(g_i) \nmid \mathbf{LM}(g_j)$  holds for all  $g_i, g_j \in \mathcal{G}$  with  $g_i \neq g_j$ , then the definition of a Gröbner basis entails that any proper subset of  $\mathcal{G}$  cannot be a Gröbner basis of  $I$ . This shows that  $\mathcal{G}$  is minimal.  $\square$

**1.4.10. Corollary** Let  $I \neq 0$  be an ideal  $I$  of  $K\langle X\rangle$ . With the notation as in Theorem 1.4.6, the Gröbner basis  $\mathcal{G}$  obtained there is indeed a minimal Gröbner basis for  $I$ . Moreover, if  $G$  is any Gröbner basis of  $I$ , then

$$\Omega = \mathbf{LM}(\mathcal{G}) = \{\mathbf{LM}(g) \mid g \in \mathcal{G}\} \subseteq \mathbf{LM}(G) = \{\mathbf{LM}(g') \mid g' \in G\}.$$

Therefore, any two minimal Gröbner bases of  $I$  have the same set of leading monomials  $\Omega$ .

**Proof** This follows immediately from the definition of a Gröbner basis, the construction of  $\Omega$  and Proposition 1.4.9.  $\square$

We next introduce the notion of a reduced Gröbner basis.

**1.4.11. Definition** Let  $I \neq \{0\}$  be an ideal of  $K\langle X\rangle$  and let  $\mathcal{G}$  be a gröbner basis of  $I$ . If  $\mathcal{G}$  satisfies the following conditions:

- (1)  $\mathcal{G}$  is a minimal Gröbner basis;
- (2)  $\mathbf{LC}(g) = 1$  for all  $g \in \mathcal{G}$ ;

(3) For every  $g \in \mathcal{G}$ ,  $h = g - \mathbf{LM}(g)$  is a normal element (mod  $\mathcal{G}$ ), i.e.,  $h \in K\text{-span}\mathcal{N}(\mathcal{G})$ ,  
 then  $\mathcal{G}$  is called a *reduced Gröbner basis* of  $N$ .

**1.4.12. Proposition** Every nonzero ideal  $I$  of  $K\langle X \rangle$  has a unique reduced Gröbner basis. Therefore, two ideals  $I$  and  $J$  have the same reduced Gröbner basis if and only if  $I = J$ .

**Proof** Note that if  $\mathcal{G}$  and  $\mathcal{G}'$  are reduced Gröbner bases for  $I$ , then  $\mathbf{LM}(\mathcal{G}) = \mathbf{LM}(\mathcal{G}')$  by Corollary 1.4.10. If  $\mathbf{LM}(g_i) = \mathbf{LM}(g'_j)$  then  $g_i - g'_j \in I \cap K\text{-span}\mathcal{N}(\mathcal{G}) = I \cap K\text{-span}\mathcal{N}(\mathcal{G}')$ . It follows from Proposition 1.4.7(iv) that  $g_i = g'_j$ . Hence  $\mathcal{G} = \mathcal{G}'$ . By the uniqueness, the second assertion is clear.  $\square$

In the case that an ideal  $I$  has a finite Gröbner basis, a minimal Gröbner basis, thereby the reduced Gröbner basis for  $I$ , can be obtained in an algorithmic way. More precisely, the next proposition holds true.

**1.4.13. Proposition** Let  $I \neq \{0\}$  be an ideal of  $K\langle X \rangle$ , and let  $\mathcal{G} = \{g_1, \dots, g_m\}$  be a finite Gröbner basis of  $I$ .

(i) The subset  $\mathcal{G}_0 = \{g_i \in \mathcal{G} \mid \mathbf{LM}(g_i) \text{ is minimal in } \mathbf{LM}(\mathcal{G}) \text{ w.r.t. } \prec\}$  of  $\mathcal{G}$  forms a minimal Gröbner basis of  $I$  (see the definition of  $\prec$  given before Theorem 1.4.6). An algorithm written in pseud-code is omitted here.

(ii) With  $\mathcal{G}_0$  as in (i) above, we may assume, without loss of generality, that  $\mathcal{G}_0 = \{g_1, \dots, g_s\}$  with  $\mathbf{LC}(g_i) = 1$  for  $1 \leq i \leq s$ . Put  $\mathcal{G}_1 = \{g_2, \dots, g_s\}$  and  $h_1 = \overline{g_1}^{\mathcal{G}_1}$ . Then  $\mathbf{LM}(h_1) = \mathbf{LM}(g_1)$ . Put  $\mathcal{G}_2 = \{h_1, g_3, \dots, g_s\}$  and  $h_2 = \overline{g_2}^{\mathcal{G}_2}$ . Then  $\mathbf{LM}(h_2) = \mathbf{LM}(g_2)$ . Put  $\mathcal{G}_3 = \{h_1, h_2, g_4, \dots, g_s\}$  and  $h_3 = \overline{g_3}^{\mathcal{G}_3}$ , and so on. The last obtained  $\mathcal{G}_s$  is then the reduced Gröbner basis of  $I$ . An algorithm written in pseud-code is omitted here.

**Proof** This can be verified directly, so we leave it as an exercise.  $\square$

### Construction of Gröbner bases

Let the admissible system  $(\mathbb{B}, \prec_x)$  of  $K\langle X \rangle$  be as fixed before. A subset  $G$  of nonzero elements in  $K\langle X \rangle$  is said to be *LM-reduced* if  $\mathbf{LM}(g_i) \not\prec \mathbf{LM}(g_j)$  for all  $g_i \neq g_j$  in  $G$ . For  $f, g \in G$ , if there are monomials  $u, v \in \mathbb{B}$  such that

- (1)  $\mathbf{LM}(f)u = v\mathbf{LM}(g)$ , and  
 (2)  $\mathbf{LM}(f) \nparallel v$  and  $\mathbf{LM}(g) \nparallel u$ ,  
 then the element

$$o(f, u; v, g) = \frac{1}{\mathbf{LC}(f)}(f \cdot u) - \frac{1}{\mathbf{LC}(g)}(v \cdot g)$$

is referred to as an *overlap element* of  $f$  and  $g$ .

Obviously  $o(f, u; v, g)$  is generally not unique and it includes also the case  $f = g$ . For instance, look at the cases where  $\mathbf{LM}(f) = X_1X_2^2$  and  $\mathbf{LM}(g) = X_2^2X_1$ ;  $\mathbf{LM}(f) = X_3^4 = \mathbf{LM}(g)$ .

Let  $I = \langle G \rangle$  be an ideal of  $K\langle X \rangle$  generated by a finite subset  $G = \{g_1, \dots, g_t\}$  of nonzero elements. Then,  $G$  may be reduced to an LM-reduced subset  $G'$  in an algorithmic way as in Proposition 1.4.13, such that  $I = \langle G' \rangle$ . Thus, we may always assume that  $I$  is generated by a reduced finite subset  $G$ .

The next theorem and the following algorithm are known as the implementation of Bergman's diamond lemma ([Ber2], [Mor], [Gr]), and are practically used to check whether  $G$  is a Gröbner basis of  $I$  or not; if not, the algorithm produces a (possibly infinite) Gröbner basis for  $I$ . For the detailed proof of the theorem we refer the reader to loc. cit.

**1.4.14. Theorem** Let  $G = \{g_1, \dots, g_t\}$  be an LM-reduced subset of nonzero elements in  $K\langle X \rangle$ . Then  $G$  is a Gröbner basis for the ideal  $I = \langle G \rangle$  if and only if for each pair  $g_i, g_j \in G$ , including  $g_i = g_j$ , every overlap element  $o(g_i, u; v, g_j)$  of  $g_i$  and  $g_j$  has the property  $\overline{o(g_i, u; v, g_j)}^G = 0$ , that is,  $o(g_i, u; v, g_j)$  is reduced to 0 (mod  $G$ ).

If  $G$  is not a Gröbner basis of  $I$ , then the algorithm presented below returns a (finite or countably infinite) Gröbner basis  $\mathcal{G}$  for  $I$ .

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#### Algorithm-GB

---

INPUT:  $G_0 = \{g_1, \dots, g_t\}$   
 OUTPUT:  $\mathcal{G} = \{g_1, \dots, g_m, \dots\}$ , a Gröbner basis for  $I$   
 INITIALIZATION:  $\mathcal{G} := G_0$ ,  $O := \{o(g_i, g_j) \mid g_i, g_j \in G_0\} - \{0\}$   
 BEGIN  
   WHILE  $O \neq \emptyset$  DO  
     Choose any  $o(g_i, g_j) \in O$   
      $O := O - \{o(g_i, g_j)\}$

```

IF  $\overline{o(g_i, g_j)}^{\mathcal{G}} = r \neq 0$  THEN
   $O := \{o(g, r), o(r, g), o(r, r) \mid g \in \mathcal{G}\} - \{0\}$ 
   $\mathcal{G} := \mathcal{G} \cup \{r\}$ 
END
:

```

□

Since generally the WHILE loop in algorithm **Algorithm-GB** does not terminate after a finite number of executions, the algorithm is written without the ending statement for the WHILE loop.

### Getting PBW $K$ -bases via Gröbner bases

Concerning the first condition (S1) that a solvable polynomial algebra should satisfy, we recall from [Li2] the following result, which is a generalization of ([Gr], Proposition 2,14; [Li1], CH.III, Theorem 1.5).

**1.4.15. Proposition** ([Li2], Ch 4, Theorem 3.1) Let  $I \neq \{0\}$  be an ideal of the free  $K$ -algebra  $K\langle X \rangle = K\langle X_1, \dots, X_n \rangle$ , and  $A = K\langle X \rangle / I$ . Suppose that  $I$  contains a subset of  $\frac{n(n-1)}{2}$  elements

$$G = \{g_{ji} = X_j X_i - F_{ji} \mid F_{ji} \in K\langle X \rangle, 1 \leq i < j \leq n\}$$

such that with respect to some monomial ordering  $\prec_X$  on  $\mathbb{B}$ ,  $\mathbf{LM}(g_{ji}) = X_j X_i$  holds for all the  $g_{ji}$ . The following two statements are equivalent:

- (i)  $A$  has the PBW  $K$ -basis  $\mathcal{B} = \{\overline{X_1^{\alpha_1} X_2^{\alpha_2} \dots X_n^{\alpha_n}} \mid \alpha_j \in \mathbb{N}\}$  where each  $\overline{X_i}$  denotes the coset of  $I$  represented by  $X_i$  in  $A$ .
- (ii) Any subset  $\mathcal{G}$  of  $I$  containing  $G$  is a Gröbner basis for  $I$  with respect to  $\prec_X$ . □

**Remark** (i) Obviously, Proposition 1.4.15 holds true if we use any permutation  $\{X_{k_1}, \dots, X_{k_n}\}$  of  $\{X_1, \dots, X_n\}$  (see an example given in the end of this section). So, in what follows we conventionally use only  $\{X_1, \dots, X_n\}$ .

(ii) Let the ideal  $I$  be as in Proposition 1.4.15. If  $G = \{g_{ji} = X_j X_i - F_{ji} \mid F_{ji} \in K\langle X \rangle, 1 \leq i < j \leq n\}$  is a Gröbner basis of  $I$  such that  $\mathbf{LM}(g_{ji}) = X_j X_i$  for all the  $g_{ji}$ , then it is not difficult to see that the



reduced Gröbner basis of  $I$  is of the form

$$\mathcal{G} = \left\{ g_{ji} = X_j X_i - \sum_q \mu_q^{ji} X_1^{\alpha_{1q}} X_2^{\alpha_{2q}} \cdots X_n^{\alpha_{nq}} \mid \begin{array}{l} \mathbf{LM}(g_{ji}) = X_j X_i, \\ 1 \leq i < j \leq n \end{array} \right\}$$

where  $\mu_q^{ji} \in K$  and  $(\alpha_{1q}, \alpha_{2q}, \dots, \alpha_{nq}) \in \mathbb{N}^n$ .

### A characterization of solvable polynomial algebras

Bearing in mind Definition 1.1.3 and the remark made above, we are now able to present the main result of this section.

**1.4.16. Theorem** ([Li4], Theorem?) Let  $A = K[a_1, \dots, a_n]$  be a finitely generated algebra over the field  $K$ , and let  $K\langle X \rangle = K\langle X_1, \dots, X_n \rangle$  be the free  $K$ -algebra with the standard  $K$ -basis  $\mathbb{B} = \{1, X_{i_1} \cdots X_{i_s} \mid X_{i_j} \in X, s \geq 1\}$ . With notations as before, the following two statements are equivalent:

- (i)  $A$  is a solvable polynomial algebra in the sense of Definition 1.1.3.
- (ii)  $A \cong \overline{A} = K\langle X \rangle / I$  via the  $K$ -algebra epimorphism  $\pi_1: K\langle X \rangle \rightarrow A$  with  $\pi_1(X_i) = a_i$ ,  $1 \leq i \leq n$ ,  $I = \text{Ker} \pi_1$ , satisfying
  - (a) with respect to some monomial ordering  $\prec_X$  on  $\mathbb{B}$ , the ideal  $I$  has a finite Gröbner basis  $G$  and the reduced Gröbner basis of  $I$  is of the form

$$\mathcal{G} = \left\{ g_{ji} = X_j X_i - \lambda_{ji} X_i X_j - F_{ji} \mid \begin{array}{l} \mathbf{LM}(g_{ji}) = X_j X_i, \\ 1 \leq i < j \leq n \end{array} \right\}$$

where  $\lambda_{ji} \in K^*$ ,  $\mu_q^{ji} \in K$ , and  $F_{ji} = \sum_q \mu_q^{ji} X_1^{\alpha_{1q}} X_2^{\alpha_{2q}} \cdots X_n^{\alpha_{nq}}$  with  $(\alpha_{1q}, \alpha_{2q}, \dots, \alpha_{nq}) \in \mathbb{N}^n$ , thereby  $\mathcal{B} = \{\overline{X}_1^{\alpha_1} \overline{X}_2^{\alpha_2} \cdots \overline{X}_n^{\alpha_n} \mid \alpha_j \in \mathbb{N}\}$  forms a PBW  $K$ -basis for  $\overline{A}$ , where each  $\overline{X}_i$  denotes the coset of  $I$  represented by  $X_i$  in  $\overline{A}$ ; and

- (b) there is a monomial ordering  $\prec$  on  $\mathcal{B}$  such that  $\mathbf{LM}(\overline{F}_{ji}) \prec \overline{X}_i \overline{X}_j$  whenever  $\overline{F}_{ji} \neq 0$ , where  $\overline{F}_{ji} = \sum_q \mu_q^{ji} \overline{X}_1^{\alpha_{1q}} \overline{X}_2^{\alpha_{2q}} \cdots \overline{X}_n^{\alpha_{nq}}$ ,  $1 \leq i < j \leq n$ .

**Proof** (i)  $\Rightarrow$  (ii) Let  $\mathcal{B} = \{a^\alpha = a_1^{\alpha_1} \cdots a_n^{\alpha_n} \mid \alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n\}$  be the PBW  $K$ -basis of the solvable polynomial algebra  $A$  and  $\prec$  a monomial ordering on  $\mathcal{B}$ . By Definition 1.1.3, the generators of  $A$  satisfy the relations:

$$a_j a_i = \lambda_{ji} a_i a_j + f_{ji}, \quad 1 \leq i < j \leq n, \quad (*)$$

where  $\lambda_{ji} \in K^*$  and  $f_{ji} = \sum_q \mu_q^{ji} a^{\alpha(q)} \in K\text{-span}\mathcal{B}$  with  $\mathbf{LM}(f_{ji}) \prec a_i a_j$ . Consider in the free  $K$ -algebra  $K\langle X \rangle = K\langle X_1, \dots, X_n \rangle$  the subset

$$\mathcal{G} = \{g_{ji} = X_j X_i - \lambda_{ji} X_i X_j - F_{ji} \mid 1 \leq i < j \leq n\},$$

where if  $f_{ji} = \sum_q \mu_q^{ji} a_1^{\alpha_{1q}} a_2^{\alpha_{2q}} \dots a_n^{\alpha_{nq}}$  then  $F_{ji} = \sum_q \mu_q^{ji} X_1^{\alpha_{1q}} X_2^{\alpha_{2q}} \dots X_n^{\alpha_{nq}}$  for  $1 \leq i < j \leq n$ . We write  $J = \langle \mathcal{G} \rangle$  for the ideal of  $K\langle X \rangle$  generated by  $\mathcal{G}$  and put  $\overline{A} = K\langle X \rangle / J$ . Let  $\pi_1: K\langle X \rangle \rightarrow A$  be the  $K$ -algebra epimorphism with  $\pi_1(X_i) = a_i$ ,  $1 \leq i \leq n$ , and let  $\pi_2: K\langle X \rangle \rightarrow \overline{A}$  be the canonical algebra epimorphism. It follows from the universal property of the canonical homomorphism that there is an algebra epimorphism  $\varphi: \overline{A} \rightarrow A$  defined by  $\varphi(\overline{X}_i) = a_i$ ,  $1 \leq i \leq n$ , such that the following diagram of algebra homomorphisms is commutative:

$$\begin{array}{ccc} K\langle X \rangle & \xrightarrow{\pi_2} & \overline{A} \\ \pi_1 \downarrow & \swarrow \varphi & \varphi \circ \pi_2 = \pi_1 \\ A & & \end{array}$$

On the other hand, by the definition of each  $g_{ji}$  we see that every element  $\overline{H} \in \overline{A}$  may be written as  $\overline{H} = \sum_j \mu_j \overline{X}_1^{\beta_{1j}} \overline{X}_2^{\beta_{2j}} \dots \overline{X}_n^{\beta_{nj}}$  with  $\mu_j \in K$  and  $(\beta_{1j}, \dots, \beta_{nj}) \in \mathbb{N}^n$ , where each  $\overline{X}_i$  is the coset of  $J$  represented by  $X_i$  in  $\overline{A}$ . Noticing the relations presented in (\*), it is straightforward to check that the correspondence

$$\psi: \begin{array}{ccc} A & \longrightarrow & \overline{A} \\ \sum_i \lambda_i a_1^{\alpha_{1i}} \dots a_n^{\alpha_{ni}} & \mapsto & \sum_i \lambda_i \overline{X}_1^{\alpha_{1i}} \dots \overline{X}_n^{\alpha_{ni}} \end{array}$$

is an algebra homomorphism such that  $\varphi \circ \psi = 1_A$  and  $\psi \circ \varphi = 1_{\overline{A}}$ , where  $1_A$  and  $1_{\overline{A}}$  denote the identity maps of  $A$  and  $\overline{A}$  respectively. This shows that  $A \cong \overline{A}$ , thereby  $\text{Ker}\pi_1 = I = J$ ; moreover,  $\mathcal{B} = \{\overline{X}_1^{\alpha_1} \overline{X}_2^{\alpha_2} \dots \overline{X}_n^{\alpha_n} \mid \alpha_j \in \mathbb{N}\}$  forms a PBW  $K$ -basis for  $\overline{A}$ , and  $\prec$  is a monomial ordering on  $\mathcal{B}$ .

We next show that  $\mathcal{G}$  forms the reduced Gröbner basis for  $I$  as described in (a). To this end, we first show that the monomial ordering  $\prec$  on  $\mathcal{B}$  induces a monomial ordering  $\prec_x$  on the standard  $K$ -basis  $\mathbb{B}$  of  $K\langle X \rangle$ . For convenience, recall that we have used capital letters  $U, V, W, S, \dots$  to denote elements (monomials) in  $\mathbb{B}$ . Also let us fix a graded lexicographic

ordering  $\prec_{grlex}$  on  $\mathbb{B}$  with respect to a given positive-degree function  $d(\cdot)$  on  $K\langle X \rangle$  (see the definition given before Definition 1.4.1), such that

$$X_1 \prec_{lex} X_2 \prec_{lex} \cdots \prec_{lex} X_n.$$

Then, for  $U, V \in \mathbb{B}$  we define

$$U \prec_X V \text{ if } \begin{cases} \mathbf{LM}(\pi_1(U)) \prec \mathbf{LM}(\pi_1(V)), \\ \text{or} \\ \mathbf{LM}(\pi_1(U)) = \mathbf{LM}(\pi_1(V)) \text{ and } U \prec_{grlex} V. \end{cases}$$

Since  $A$  is a domain (Proposition 1.1.4(ii)) and  $\pi_1$  is an algebra homomorphism with  $\pi_1(X_i) = a_i$  for  $1 \leq i \leq n$ , it follows that  $\mathbf{LM}(\pi_1(W)) \neq 0$  for all  $W \in \mathbb{B}$ . We also note from Proposition 1.1.4(i) that if  $f, g \in A$  are nonzero elements, then  $\mathbf{LM}(fg) = \mathbf{LM}(\mathbf{LM}(f)\mathbf{LM}(g))$ . Thus, if  $U, V, W \in \mathbb{B}$  and  $U \prec_X V$  subject to  $\mathbf{LM}(\pi_1(U)) \prec \mathbf{LM}(\pi_1(V))$ , then

$$\begin{aligned} \mathbf{LM}(\pi_1(WU)) &= \mathbf{LM}(\mathbf{LM}(\pi_1(W))\mathbf{LM}(\pi_1(U))) \\ &\prec \mathbf{LM}(\mathbf{LM}(\pi_1(W))\mathbf{LM}(\pi_1(V))) \\ &= \mathbf{LM}(\pi_1(WV)) \end{aligned}$$

implies  $WU \prec_X WV$ ; if  $U \prec_X V$  subject to  $\mathbf{LM}(\pi_1(U)) = \mathbf{LM}(\pi_1(V))$  and  $U \prec_{grlex} V$ , then

$$\begin{aligned} \mathbf{LM}(\pi_1(WU)) &= \mathbf{LM}(\mathbf{LM}(\pi_1(W))\mathbf{LM}(\pi_1(U))) \\ &= \mathbf{LM}(\mathbf{LM}(\pi_1(W))\mathbf{LM}(\pi_1(V))) \\ &= \mathbf{LM}(\pi_1(WV)) \end{aligned}$$

and  $WU \prec_{grlex} WV$  implies  $WU \prec_X WV$ . Similarly, if  $U \prec_X V$  then  $US \prec_X VS$  for all  $S \in \mathbb{B}$ . Moreover, if  $W, U, V, S \in \mathbb{B}$ ,  $W \neq V$ , such that  $W = UVS$ , then  $\mathbf{LM}(\pi_1(W)) = \mathbf{LM}(\pi_1(UVS))$  and clearly  $V \prec_{grlex} W$ , thereby  $V \prec_X W$ . Since  $\prec$  is a well-ordering on  $\mathcal{B}$  and  $\prec_{grlex}$  is a well-ordering on  $\mathbb{B}$ , the above argument shows that  $\prec_X$  is a monomial ordering on  $\mathbb{B}$ . With this monomial ordering  $\prec_X$  in hand, by the definition of  $F_{ji}$  we see that  $\mathbf{LM}(F_{ji}) \prec_X X_i X_j$ . Furthermore, since  $\mathbf{LM}(\pi_1(X_j X_i)) = a_i a_j = \mathbf{LM}(\pi_1(X_i X_j))$  and  $X_i X_j \prec_{grlex} X_j X_i$ , we see that  $X_i X_j \prec_X X_j X_i$ . It follows that  $\mathbf{LM}(g_{ji}) = X_j X_i$  for  $1 \leq i < j \leq n$ . Now, by Proposition 1.4.14 we conclude that  $\mathcal{G}$  forms a Gröbner basis for  $I$  with respect to  $\prec_X$ . Finally, by the definition of  $\mathcal{G}$ , it is clear that  $\mathcal{G}$  is the reduced Gröbner basis of  $I$ , as desired.

(ii)  $\Rightarrow$  (i) Note that (a) + (b) tells us that the generators of  $\overline{A}$  satisfy the relations  $\overline{X}_j \overline{X}_i = \lambda_{ji} \overline{X}_i \overline{X}_j + \overline{F}_{ji}$ ,  $1 \leq i < j \leq n$ , and that if  $\overline{F}_{ji} \neq 0$  then  $\mathbf{LM}(\overline{F}_{ji}) \prec \overline{X}_i \overline{X}_j$  with respect to the given monomial ordering  $\prec$  on  $\mathcal{B}$ . It follows that  $\overline{A}$  and hence  $A$  is a solvable polynomial algebra in the sense of Definition 1.1.3.  $\square$

**Remark** The monomial ordering  $\prec_x$  we defined in the proof of Theorem 1.4.16 is a modification of the *lexicographic extension* defined in [EPS]. But our definition of  $\prec_x$  involves a graded monomial ordering  $\prec_{grlex}$  on the standard  $K$ -basis  $\mathbb{B}$  of the free  $K$ -algebra  $K\langle X \rangle = K\langle X_1, \dots, X_n \rangle$ . The reason is that *the monomial ordering  $\prec_x$  on  $\mathbb{B}$  must be compatible with the usual rule of division*, namely,  $W, U, V, S \in \mathbb{B}$ ,  $W \neq V$ , and  $W = UVS$  implies  $V \prec_x W$ . While it is clear that if we use any lexicographic ordering  $\prec_{lex}$  in the definition of  $\prec_x$ , then this rule will not work in general.

We end this section by an example illustrating Theorem 1.4.16, in particular, illustrating that the monomial ordering  $\prec_x$  used in the condition (a) and the monomial ordering  $\prec$  used in the condition (b) may be mutually independent, namely  $\prec$  may not necessarily be the restriction of  $\prec_x$  on  $\mathcal{B}$ , and the choice of  $\prec$  is indeed quite flexible.

**Example** (1) Considering the positive-degree function  $d(\ )$  on the free  $K$ -algebra  $K\langle X \rangle = K\langle X_1, X_2, X_3 \rangle$  such that  $d(X_1) = 2$ ,  $d(X_2) = 1$ , and  $d(X_3) = 4$ , let  $I$  be the ideal of  $K\langle X \rangle$  generated by the elements

$$\begin{aligned} g_1 &= X_1 X_2 - X_2 X_1, \\ g_2 &= X_3 X_1 - \lambda X_1 X_3 - \mu X_3 X_2^2 - f(X_2), \\ g_3 &= X_3 X_2 - X_2 X_3, \end{aligned}$$

where  $\lambda \in K^*$ ,  $\mu \in K$ ,  $f(X_2)$  is a polynomial in  $X_2$  which has degree  $\leq 6$ , or  $f(X_2) = 0$ . The following properties hold.

- (1) If we use the graded lexicographic ordering  $X_2 \prec_{grlex} X_1 \prec_{grlex} X_3$  on  $K\langle X \rangle$ , then the three generators have the leading monomials  $\mathbf{LM}(g_1) = X_1 X_2$ ,  $\mathbf{LM}(g_2) = X_3 X_1$ , and  $\mathbf{LM}(g_3) = X_3 X_2$ . It is an exercise to verify that  $\mathcal{G} = \{g_1, g_2, g_3\}$  forms a Gröbner basis for  $I$  with respect to  $\prec_{grlex}$ .
- (2) With respect to the fixed  $\prec_{grlex}$  in (1), the reduced Gröbner basis  $\mathcal{G}'$

of  $I$  consists of

$$\begin{aligned} g_1 &= X_1X_2 - X_2X_1, \\ g_2 &= X_3X_1 - \lambda X_1X_3 - \mu X_2^2X_3 - f(X_2), \\ g_3 &= X_3X_2 - X_2X_3, \end{aligned}$$

(3) Writing  $A = K[a_1, a_2, a_3]$  for the quotient algebra  $K\langle X \rangle/I$ , where  $a_1$ ,  $a_2$  and  $a_3$  denote the cosets  $X_1 + I$ ,  $X_2 + I$  and  $X_3 + I$  in  $K\langle X \rangle/I$  respectively, it follows that  $A$  has the PBW basis  $\mathcal{B} = \{a^\alpha = a_2^{\alpha_2} a_1^{\alpha_1} a_3^{\alpha_3} \mid \alpha = (\alpha_2, \alpha_1, \alpha_3) \in \mathbb{N}^3\}$ . Noticing that  $a_2a_1 = a_1a_2$ , it is clear that  $\mathcal{B}' = \{a^\alpha = a_1^{\alpha_1} a_2^{\alpha_2} a_3^{\alpha_3} \mid \alpha = (\alpha_1, \alpha_2, \alpha_3) \in \mathbb{N}^3\}$  is also a PBW basis for  $A$ . Since  $a_3a_1 = \lambda a_1a_3 + \mu a_2^2a_3 + f(a_2)$ , where  $f(a_2) \in K\text{-span}\{1, a_2, a_2^2, \dots, a_2^6\}$ , we see that  $A$  has the monomial ordering  $\prec_{lex}$  on  $\mathcal{B}'$  such that  $a_3 \prec_{lex} a_2 \prec_{lex} a_1$  and  $\mathbf{LM}(\mu a_2^2a_3 + f(a_2)) \prec_{lex} a_1a_3$ , thereby  $A$  is turned into a solvable polynomial algebra with respect to  $\prec_{lex}$ .

Moreover, it is also an exercise to check that if we use the positive-degree function  $d(\cdot)$  on  $\mathcal{B}'$  such that  $d(a_1) = 2$ ,  $d(a_2) = 1$ , and  $d(a_3) = 4$ , then  $A$  has another monomial ordering on  $\mathcal{B}'$ , namely the graded lexicographic ordering  $\prec_{grlex}$  such that  $a_3 \prec_{grlex} a_2 \prec_{grlex} a_1$  and  $\mathbf{LM}(\mu a_2^2a_3 + f(a_2)) \prec_{grlex} a_1a_3$ , thereby  $A$  is turned into a solvable polynomial algebra with respect to  $\prec_{grlex}$ .

## 2. Left Gröbner Bases for Modules

Based on the theory of left Gröbner bases for left ideals of solvable polynomial algebras presented in Chapter 1, in this chapter we introduce *left Gröbner bases* for submodules of *free left modules* over solvable polynomial algebras. More precisely, let  $A = K[a_1, \dots, a_n]$  be a solvable polynomial algebra with admissible system  $(\mathcal{B}, \prec)$  as described in Chapter 1. In the first section, we define left monomial orderings, especially the graded left monomial orderings and Schreyer orderings, on free  $A$ -modules. In Section 2, we introduce left Gröbner bases, minimal left Gröbner bases and reduced left Gröbner bases for submodules of free left  $A$ -modules via a left division algorithm, and we discuss some basic properties of left Gröbner bases, in particular, we show that every submodule of a free left  $A$ -module  $L = \oplus_{i=1}^s Ae_i$  has a finite left Gröbner basis. In Section 3, we define left S-polynomials for pairs of elements in free  $A$ -modules, and we show that a noncommutative version of Buchberger's criterion holds true for submodules of free  $A$ -modules, and that a noncommutative version of the Buchberger algorithm works effectively for computing finite left Gröbner bases of submodules in free  $A$ -modules.

The main references of this chapter are [AL2], [Eis], [KR1], [K-RW], [Li1], [Lev], [DGPS].

Throughout this chapter, modules are meant left  $A$ -modules, and all notions and notations used in Chapter 1 are maintained.

## 2.1. Left Monomial Orderings on Free Modules

Let  $A = K[a_1, \dots, a_n]$  be a solvable polynomial algebra with admissible system  $(\mathcal{B}, \prec)$  in the sense of Definition 1.1.3, where  $\mathcal{B} = \{a^\alpha = a_1^{\alpha_1} \cdots a_n^{\alpha_n} \mid \alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n\}$  is the PBW  $K$ -basis of  $A$  and  $\prec$  is a monomial ordering on  $\mathcal{B}$ , and let  $L = \bigoplus_{i=1}^s A e_i$  be a free  $A$ -module with the  $A$ -basis  $\{e_1, \dots, e_s\}$ . Then  $L$  has the  $K$ -basis

$$\mathcal{B}(e) = \{a^\alpha e_i \mid a^\alpha \in \mathcal{B}, 1 \leq i \leq s\}.$$

For convenience, elements of  $\mathcal{B}(e)$  are also referred to as *monomials* in  $L$ .

If  $\prec_e$  is a total ordering on  $\mathcal{B}(e)$ , and if  $\xi = \sum_{j=1}^m \lambda_j a^{\alpha(j)} e_{i_j} \in L$ , where  $\lambda_j \in K^*$  and  $\alpha(j) = (\alpha_{j1}, \dots, \alpha_{jn}) \in \mathbb{N}^n$ , such that

$$a^{\alpha(1)} e_{i_1} \prec_e a^{\alpha(2)} e_{i_2} \prec_e \cdots \prec_e a^{\alpha(m)} e_{i_m},$$

then by **LM**( $\xi$ ) we denote the *leading monomial*  $a^{\alpha(m)} e_{i_m}$  of  $\xi$ , by **LC**( $\xi$ ) we denote the *leading coefficient*  $\lambda_m$  of  $\xi$ , and by **LT**( $\xi$ ) we denote the *leading term*  $\lambda_m a^{\alpha(m)} e_{i_m}$  of  $f$ .

**2.1.1. Definition** With respect to the given monomial ordering  $\prec$  on  $\mathcal{B}$ , a total ordering  $\prec_e$  on  $\mathcal{B}(e)$  is called a *left monomial ordering* if the following two conditions are satisfied:

- (1)  $a^\alpha e_i \prec_e a^\beta e_j$  implies **LM**( $a^\gamma a^\alpha e_i$ )  $\prec_e$  **LM**( $a^\gamma a^\beta e_j$ ) for all  $a^\alpha e_i, a^\beta e_j \in \mathcal{B}(e)$ ,  $a^\gamma \in \mathcal{B}$ ;
- (2)  $a^\beta \prec a^\alpha$  implies  $a^\alpha e_i \prec_e a^\beta e_i$  for all  $a^\alpha, a^\beta \in \mathcal{B}$  and  $1 \leq i \leq s$ .

If  $\prec_e$  is a left monomial ordering on  $\mathcal{B}(e)$ , then we also say that  $\prec_e$  is a left monomial ordering on the free module  $L$ , and the data  $(\mathcal{B}(e), \prec_e)$  is referred to as a *left admissible system* of  $L$ .

By referring to Proposition 1.1.4, we record two easy but useful facts on a left monomial ordering  $\prec_e$  on  $\mathcal{B}(e)$ , as follows.

- 2.1.2. Lemma** (i) Every left monomial ordering  $\prec_e$  on  $\mathcal{B}(e)$  is a well-ordering, i.e., every nonempty subset of  $\mathcal{B}(e)$  has a minimal element.  
(ii) If  $f \in A$  with **LM**( $f$ ) =  $a^\gamma$  and  $\xi \in L$  with **LM**( $\xi$ ) =  $a^\alpha e_i$ , then

$$\mathbf{LM}(f\xi) = \mathbf{LM}(\mathbf{LM}(f)\mathbf{LM}(\xi)) = \mathbf{LM}(a^\gamma a^\alpha e_i) = a^{\gamma+\alpha} e_i.$$

□

Actually as in the commutative case ([AL2], [Eis], [KR1]), any monomial ordering  $\prec$  on  $\mathcal{B}$  may induce two left monomial orderings on  $\mathcal{B}(e)$ :

$$\begin{aligned} (\textbf{TOP ordering}) \quad & a^\alpha e_i \prec_e a^\beta e_j \Leftrightarrow a^\alpha \prec a^\beta, \text{ or } a^\alpha = a^\beta \text{ and } i < j; \\ (\textbf{POT ordering}) \quad & a^\alpha e_i \prec_e a^\beta e_j \Leftrightarrow i < j, \text{ or } i = j \text{ and } a^\alpha \prec a^\beta. \end{aligned}$$

Let  $d(\cdot)$  be a positive-degree function on  $A$  (see Section 1 of Chapter 1) such that  $d(a_i) = m_i > 0$ ,  $1 \leq i \leq n$ , and let  $(b_1, \dots, b_s) \in \mathbb{N}^s$  be any fixed  $s$ -tuple. Then, by assigning  $e_j$  the degree  $b_j$ ,  $1 \leq j \leq s$ , every monomial  $a^\alpha e_j$  in the  $K$ -basis  $\mathcal{B}(e)$  of  $L$  is endowed with the degree  $d(a^\alpha) + b_j$ . Similar to Definition 1.1.2, if a left monomial ordering  $\prec_e$  on  $\mathcal{B}(e)$  satisfies

$$a^\alpha e_i \prec_e a^\beta e_j \text{ implies } d(a^\alpha) + b_i \leq d(a^\beta) + b_j,$$

then we call it a *graded left monomial ordering* on  $\mathcal{B}(e)$ .

We refer the reader to Chapter 4 and Chapter 5 for examples of graded left monomial orderings, also for the reason that a graded left monomial ordering is defined in such a way.

Let  $\prec_e$  be a left monomial ordering on  $\mathcal{B}(e)$ , and let  $L_1 = \bigoplus_{i=1}^m A\varepsilon_i$  be another free  $A$ -module with the  $A$ -basis  $\{\varepsilon_1, \dots, \varepsilon_m\}$ . Then, as in the commutative case ([AL2], [Eis], [KR1]), for any given finite subset  $G = \{g_1, \dots, g_m\} \subset L$ , an ordering on the  $K$ -basis  $\mathcal{B}(\varepsilon) = \{a^\alpha \varepsilon_i \mid a^\alpha \in \mathcal{B}, 1 \leq i \leq m\}$  of  $L_1$  can be defined subject to the rule: for  $a^\alpha \varepsilon_i, a^\beta \varepsilon_j \in \mathcal{B}(\varepsilon)$ ,

$$a^\alpha \varepsilon_i \prec_{s-\varepsilon} a^\beta \varepsilon_j \Leftrightarrow \begin{cases} \mathbf{LM}(a^\alpha g_i) \prec_e \mathbf{LM}(a^\beta g_j), \\ \text{or} \\ \mathbf{LM}(a^\alpha g_i) = \mathbf{LM}(a^\beta g_j) \text{ and } i < j, \end{cases}$$

It is an exercise to check that this ordering is a left monomial ordering on  $\mathcal{B}(\varepsilon)$ .  $\prec_{s-\varepsilon}$  is usually referred to as the *Schreyer ordering* induced by  $G$  with respect to  $\prec_e$ .

## 2.2. Left Gröbner Bases of Submodules

Let  $A = K[a_1, \dots, a_n]$  be a solvable polynomial algebra with admissible system  $(\mathcal{B}, \prec)$ , and let  $L = \bigoplus_{i=1}^s A e_i$  be a free  $A$ -module with left admissible system  $(\mathcal{B}(e), \prec_e)$ . In this section we introduce left Gröbner bases for



submodules of  $L$  via a left division algorithm in  $L$ , and we record some basic facts determined by left Gröbner bases. Moreover, minimal left Gröbner bases and reduced left Gröbner bases are discussed. Finally, we show that every nonzero submodule of  $L$  has a finite left Gröbner basis.

### Left division algorithm

Let  $a^\alpha e_i, a^\beta e_j \in \mathcal{B}(e)$ , where  $\alpha = (\alpha_1, \dots, \alpha_n), \beta = (\beta_1, \dots, \beta_n) \in \mathbb{N}^n$ . Then, the left division of monomials in  $\mathcal{B}$  we introduced in (Section 2 of Chapter 1) gives rise to a left division for monomials in  $\mathcal{B}(e)$ , that is, we say that  $a^\alpha e_i$  divides  $a^\beta e_j$  from left side, denoted  $a^\alpha e_i \mid_L a^\beta e_j$ , if  $i = j$  and there is some  $a^\gamma \in \mathcal{B}$  such that

$$a^\beta e_i = \mathbf{LM}(a^\gamma a^\alpha e_i).$$

It follows from Lemma 2.1.2(ii) that the division defined above is implementable.

Let  $\Xi$  be a nonempty subset of  $L$ . Then the division of monomials defined above yields a subset of  $\mathcal{B}(e)$ :

$$\mathcal{N}(\Xi) = \{a^\alpha e_i \in \mathcal{B}(e) \mid \mathbf{LM}(\xi) \nmid_L a^\alpha e_i, \xi \in \Xi\}.$$

If  $\Xi = \{\xi\}$  consists of a single element  $\xi$ , then we simply write  $\mathcal{N}(\xi)$  in place of  $\mathcal{N}(\Xi)$ . Also we write  $K\text{-span}\mathcal{N}(\Xi)$  for the  $K$ -subspace of  $L$  spanned by  $\mathcal{N}(\Xi)$ .

**2.2.1. Definition** Elements of  $\mathcal{N}(\Xi)$  are referred to as *normal monomials* (mod  $\Xi$ ). Elements of  $K\text{-span}\mathcal{N}(\Xi)$  are referred to as *normal elements* (mod  $\Xi$ ).

In view of Lemma 2.1.2(ii), the left division we defined for monomials in  $\mathcal{B}(e)$  can naturally be used to define a left division procedure for elements in  $L$ . More precisely, let  $\xi, \zeta \in L$  with  $\mathbf{LC}(\xi) = \mu \neq 0$ ,  $\mathbf{LC}(\zeta) = \lambda \neq 0$ . If  $\mathbf{LM}(\zeta) \mid_L \mathbf{LM}(\xi)$ , i.e., there exists  $a^\alpha \in \mathcal{B}$  such that  $\mathbf{LM}(\xi) = \mathbf{LM}(a^\alpha \mathbf{LM}(\zeta))$ , then put  $\xi_1 = \xi - \lambda^{-1} \mu a^\alpha \zeta$ ; otherwise, put  $\xi_1 = \xi - \mathbf{LT}(\xi)$ . Note that in both cases we have  $\xi_1 = 0$ , or  $\xi_1 \neq 0$  and  $\mathbf{LM}(\xi_1) \prec_e \mathbf{LM}(\xi)$ . At this stage, let us refer to such a procedure of canceling the leading term of  $\xi$  as the *left division procedure by  $\zeta$* . With

$\xi := \xi_1 \neq 0$ , we can repeat the left division procedure by  $\zeta$  and so on. This returns successively a descending sequence

$$\mathbf{LM}(\xi_{i+1}) \prec_e \mathbf{LM}(\xi_i) \prec_e \cdots \prec_e \mathbf{LM}(\xi_1) \prec_e \mathbf{LM}(\xi), \quad i \geq 2.$$

Since  $\prec_e$  is a well-ordering, it follows that such a division procedure terminates after a finite number of repetitions, and consequently  $\xi$  is expressed as

$$\xi = q\zeta + \eta,$$

where  $q \in A$  and  $\eta \in K\text{-span}\mathcal{N}(\zeta)$ , i.e.,  $\eta$  is normal (mod  $\zeta$ ), such that either  $\mathbf{LM}(\xi) = \mathbf{LM}(q\zeta)$  or  $\mathbf{LM}(\xi) = \mathbf{LM}(\eta)$ .

Actually as in (Section 2 of Chapter 1), the left division procedure demonstrated above can be extended to a left division procedure by a finite subset  $\Xi = \{\xi_1, \dots, \xi_s\}$  in  $L$ , which yields the following left division algorithm:

---

**Algorithm-DIV-L**

---

INPUT:  $\xi, \Xi = \{\xi_1, \dots, \xi_s\}$  with  $\xi_i \neq 0$  ( $1 \leq i \leq s$ )

OUTPUT:  $q_1, \dots, q_s \in A, \eta \in K\text{-span}\mathcal{N}(\Xi)$ , such that

$$\xi = \sum_{i=1}^s q_i \xi_i + \eta, \quad \mathbf{LM}(q_i \xi_i) \preceq_e \mathbf{LM}(\xi) \text{ for } q_i \neq 0, \\ \mathbf{LM}(\eta) \preceq_e \mathbf{LM}(\xi) \text{ if } \eta \neq 0$$

INITIALIZATION:  $q_1 := 0, q_2 := 0, \dots, q_s := 0; \eta := 0; \omega := \xi$

BEGIN

  WHILE  $\omega \neq 0$  DO

    IF there exist  $i$  and  $a^{\alpha(i)} \in \mathcal{B}$  such that

$\mathbf{LM}(\omega) = \mathbf{LM}(a^{\alpha(i)} \mathbf{LM}(\xi_i))$  THEN

      choose  $i$  least such that  $\mathbf{LM}(\omega) = \mathbf{LM}(a^{\alpha(i)} \mathbf{LM}(\xi_i))$

$$q_i := q_i + \mathbf{LC}(\xi_i)^{-1} \mathbf{LC}(\omega) a^{\alpha(i)}$$

$$\omega := \omega - \mathbf{LC}(\xi_i)^{-1} \mathbf{LC}(\omega) a^{\alpha(i)} \xi_i$$

    ELSE

$$\eta := \eta + \mathbf{LT}(\omega)$$

$$\omega := \omega - \mathbf{LT}(\omega)$$

    END

  END

END

---

**2.2.2. Definition** The element  $\eta$  obtained in **Algorithm-DIV-L** is called a *remainder* of  $\xi$  on left division by  $\Xi$ , and is denoted by  $\bar{\xi}^\Xi$ , i.e.,  $\bar{\xi}^\Xi = \eta$ . If  $\bar{\xi}^\Xi = 0$ , then we say that  $\xi$  is *reduced to 0 (mod  $\Xi$ )*.

**Remark** For the reason that the element  $\eta$  obtained in **Algorithm-DIV-L** depends on how an order of elements in  $\Xi$  is arranged, we used the phrase “a remainder” in the above definition. In other words, the element  $\eta$  may be different if a different order for elements in  $\Xi$  is given.

Summing up, we have reached the following

**2.2.3. Theorem** Given a set of nonzero elements  $\Xi = \{\xi_1, \dots, \xi_s\}$  and  $\xi$  in  $L$ , the **Algorithm-DIV-L** produces elements  $q_1, \dots, q_s \in A$  and  $\eta \in K\text{-span}\mathcal{N}(\Xi)$ , such that  $\xi = \sum_{i=1}^s q_i \xi_i + \eta$  and  $\mathbf{LM}(q_i \xi_i) \preceq_e \mathbf{LM}(\xi)$  whenever  $q_i \neq 0$ ,  $\mathbf{LM}(\eta) \preceq_e \mathbf{LM}(\xi)$  if  $\eta \neq 0$ .  $\square$

### Left Gröbner bases

It is theoretically correct that the left division procedure by using a finite subset  $\Xi$  of nonzero elements in  $L$  can be extended to a left division procedure by means of an *arbitrary proper subset*  $G$  of nonzero elements, for, it is a true statement that if  $\xi \in L$  with  $\mathbf{LM}(\xi) \neq 0$ , then either there exists  $g \in G$  such that  $\mathbf{LM}(g)|_L \mathbf{LM}(\xi)$  or such  $g$  does not exist. This leads to the following

**2.2.4. Proposition** Let  $N$  be a submodule of  $L$  and  $G$  a proper subset of nonzero elements in  $N$ . The following two statements are equivalent:

- (i) If  $\xi \in N$  and  $\xi \neq 0$ , then there exists  $g \in G$  such that  $\mathbf{LM}(g)|_L \mathbf{LM}(\xi)$ .
- (ii) Every nonzero  $\xi \in N$  has a representation  $\xi = \sum_{j=1}^s q_j g_{i_j}$  with  $q_j \in A$  and  $g_{i_j} \in G$ , such that  $\mathbf{LM}(q_j g_{i_j}) \preceq_e \mathbf{LM}(\xi)$  whenever  $q_j \neq 0$ .  $\square$

**2.2.5. Definition** Let  $N$  be a submodule of  $L$ . With respect to a given left monomial ordering  $\prec_e$  on  $\mathcal{B}(e)$ , a proper subset  $\mathcal{G}$  of nonzero elements in  $N$  is said to be a *left Gröbner basis* of  $N$  if  $\mathcal{G}$  satisfies one of the equivalent conditions of Proposition 2.2.4.

If  $\mathcal{G}$  is a left Gröbner basis of  $N$ , then the expression  $\xi = \sum_{j=1}^s q_j g_{i_j}$  appeared in Proposition 2.2.4(ii) is called a *left Gröbner representation* of  $\xi$ .

Clearly, if  $N$  is a submodule  $L$  and  $N$  has a left Gröbner basis  $\mathcal{G}$ , then  $\mathcal{G}$  is certainly a generating set of  $N$ , i.e.,  $N = \sum_{g \in \mathcal{G}} Ag$ . But the converse is not necessarily true. For instance, consider the solvable polynomial algebra  $A = \mathbb{C}[a_1, a_2, a_3]$  generated by  $\{a_1, a_2, a_3\}$  subject to the relations

$$a_2a_1 = 3a_1a_2, \quad a_3a_1 = a_1a_3, \quad a_3a_2 = 5a_2a_3,$$

and let  $L = Ae_1 \oplus Ae_2 \oplus Ae_3$  be the free  $A$ -module with the  $A$ -basis  $\{e_1, e_2, e_3\}$ . Then, with  $g_1 = a_1^2a_2e_1 - a_3e_3$ ,  $g_2 = a_2e_1$ , and  $N = Ag_1 + Ag_2$ , we have  $a_3e_e \in N$ . Hence the set  $G = \{g_1, g_2\}$  is not a left Gröbner basis with respect to any given monomial ordering  $\prec_e$  on  $\mathcal{B}(e)$  such that  $\mathbf{LM}(g_1) = a_1^2a_2e_1$ ,  $\mathbf{LM}(g_2) = a_2e_1$ .

### The existence of left Gröbner bases

Noticing  $1 \in \mathcal{B}$ , if we define on  $\mathcal{B}(e)$  the ordering:

$$a^\alpha e_i \prec'_e a^\beta e_j \Leftrightarrow a^\alpha e_i|_L a^\beta e_j, \quad a^\alpha e_i, a^\beta e_j \text{ in } \mathcal{B}(e),$$

then, by Definition 2.1.1, Lemma 2.1.2(ii) and the divisibility defined for monomials in  $\mathcal{B}(e)$ , it is an easy exercise to check that  $\prec'$  is reflexive, anti-symmetric, transitive, and moreover,

$$a^\alpha e_i \prec'_e a^\beta e_j \text{ implies } a^\alpha e_i \prec_e a^\beta e_j.$$

Since the given left monomial ordering  $\prec_e$  is a well-ordering on  $\mathcal{B}(e)$ , it follows that every nonempty subset of  $\mathcal{B}(e)$  has a minimal element with respect to the ordering  $\prec'_e$  on  $\mathcal{B}(e)$ .

Now, let  $N \neq \{0\}$  be a submodule of  $L$ ,  $\mathbf{LM}(N) = \{\mathbf{LM}(\xi) \mid \xi \in N\}$ , and  $\Omega = \{a^\alpha e_i \in \mathbf{LM}(N) \mid a^\alpha e_i \text{ is minimal in } \mathbf{LM}(N) \text{ w.r.t. } \prec'_e\}$ .

**2.2.6. Theorem** With the notation as above, the following statements hold.

- (i)  $\Omega \neq \emptyset$  and  $\Omega$  is a proper subset of  $\mathbf{LM}(N)$ .
- (ii) Let  $\mathcal{G} = \{g \in N \mid \mathbf{LM}(g) \in \Omega\}$ . Then  $\mathcal{G}$  is a left Gröbner basis of  $N$ .

**Proof** (i) That  $\Omega \neq \emptyset$  follows from  $N \neq \{0\}$  and the remark about  $\prec'_e$  we made above. Since  $N$  is a submodule of  $L$  and  $A$  is a domain (Proposition 1.1.4), if  $\xi \in N$  and  $\xi \neq 0$ , then  $h\xi \in N$  for all nonzero  $h \in A$ . It follows from Lemma 2.1.2(ii) that  $0 \neq \mathbf{LM}(h\xi) \in \mathbf{LM}(N)$  and  $\mathbf{LM}(\xi)|_L \mathbf{LM}(h\xi)$ ,

i.e.,  $\mathbf{LM}(\xi) \prec'_e \mathbf{LM}(h\xi)$  in  $\mathbf{LM}(N)$ . It is clear that if  $\mathbf{LM}(h) \neq 1$ , then  $\mathbf{LM}(h\xi) \neq \mathbf{LM}(\xi)$ . This shows that  $\Omega$  is a proper subset of  $\mathbf{LM}(N)$ .

(ii) By the definition of  $\prec'_e$  and the remark we made above the theorem, if  $\xi \in N$  with  $\mathbf{LM}(\xi) \neq 0$  and  $\mathbf{LM}(\xi) \notin \Omega$ , then there is some  $g \in \mathcal{G}$  such that  $\mathbf{LM}(g)|_{\mathbf{L}} \mathbf{LM}(\xi)$ . Hence the selected  $\mathcal{G}$  is a left Gröbner basis for  $N$ .  $\square$

Since every solvable polynomial algebra  $A$  is (left and right) Noetherian by Corollary 1.3.3, the free module  $L = \oplus_{i=1}^s Ae_i$  is a Noetherian left  $A$ -module, thereby every submodule  $N$  of  $L$  is finitely generated. In the next section we will show that if a finite generating set  $\Xi = \{\xi_1, \dots, \xi_t\}$  of  $N$  is given, then a *finite* left Gröbner basis  $\mathcal{G}$  for  $N$  can be produced by means of a noncommutative Buchberger algorithm.

### Basic facts determined by left Gröbner bases

By referring to Lemma 2.1.2 Theorem 2.2.3 and Proposition 2.2.4, the foregoing discussion allows us to summarize some basic facts determined by left Gröbner bases, of which the detailed proof is left as an exercise.

**2.2.7. Proposition** Let  $N$  be a submodule of the free  $A$ -module  $L = \oplus_{i=1}^s Ae_i$ , and let  $\mathcal{G}$  be a left Gröbner basis of  $N$ . Then the following statements hold.

- (i) If  $\xi \in N$  and  $\xi \neq 0$ , then  $\mathbf{LM}(\xi) = \mathbf{LM}(qg)$  for some  $q \in A$  and  $g \in \mathcal{G}$ .
- (ii) If  $\xi \in L$  and  $\xi \neq 0$ , then  $\xi$  has a unique remainder  $\bar{\xi}^{\mathcal{G}}$  on division by  $\mathcal{G}$ .
- (iii) If  $\xi \in L$  and  $\xi \neq 0$ , then  $\xi \in N$  if and only if  $\bar{\xi}^{\mathcal{G}} = 0$ . Hence the membership problem for submodules of  $L$  can be solved by using left Gröbner bases.
- (iv) As a  $K$ -vector space,  $L$  has the decomposition

$$L = N \oplus K\text{-span}\mathcal{N}(\mathcal{G}).$$

- (v) As a  $K$ -vector space,  $L/N$  has the  $K$ -basis

$$\overline{\mathcal{N}(\mathcal{G})} = \{\overline{a^\alpha e_i} \mid a^\alpha e_i \in \mathcal{N}(\mathcal{G})\},$$

where  $\overline{a^\alpha e_i}$  denotes the coset represented by  $a^\alpha e_i$  in  $L/N$ .

- (vi)  $\mathcal{N}(\mathcal{G})$  is a finite set, or equivalently,  $\dim_K L/N < \infty$ , if and only if for  $j = 1, \dots, n$  and each  $i = 1, \dots, s$ , there exists  $g_{j_i} \in \mathcal{G}$  such that  $\mathbf{LM}(g_{j_i}) = a_j^{m_j} e_i$ , where  $m_j \in \mathbb{N}$ .  $\square$

### Minimal and reduced left Gröbner bases

**2.2.8. Definition** Let  $N \neq \{0\}$  be a submodule of  $L$  and let  $\mathcal{G}$  be a left gröbner basis of  $N$ . If any proper subset of  $\mathcal{G}$  cannot be a left Gröbner basis of  $N$ , then  $\mathcal{G}$  is called a *minimal left Gröbner basis* of  $N$ .

**2.2.9. Proposition** Let  $N \neq \{0\}$  be a submodule of  $L$ . A left Gröbner basis  $\mathcal{G}$  of  $N$  is minimal if and only if  $\mathbf{LM}(g_i) \not\ll_L \mathbf{LM}(g_j)$  for all  $g_i, g_j \in \mathcal{G}$  with  $g_i \neq g_j$ .

**Proof** Suppose that  $\mathcal{G}$  is minimal. If there were  $g_i \neq g_j$  in  $\mathcal{G}$  such that  $\mathbf{LM}(g_i) \ll_L \mathbf{LM}(g_j)$ , then since the left division is transitive, the proper subset  $\mathcal{G}' = \mathcal{G} - \{g_j\}$  of  $\mathcal{G}$  would form a left Gröbner basis of  $N$ . This contradicts the minimality of  $\mathcal{G}$ .

Conversely, if the condition  $\mathbf{LM}(g_i) \not\ll_L \mathbf{LM}(g_j)$  holds for all  $g_i, g_j \in \mathcal{G}$  with  $g_i \neq g_j$ , then the definition of a left Gröbner basis entails that any proper subset of  $\mathcal{G}$  cannot be a left Gröbner basis of  $N$ . This shows that  $\mathcal{G}$  is minimal.  $\square$

**2.2.10. Corollary** Let  $N \neq \{0\}$  be a submodule of  $L$ , and let the notation be as in Theorem 2.2.6.

- (i) The left Gröbner basis  $\mathcal{G}$  obtained there is indeed a minimal left Gröbner basis for  $N$ .
- (ii) If  $G$  is any left Gröbner basis of  $N$ , then

$$\Omega = \mathbf{LM}(\mathcal{G}) = \{\mathbf{LM}(g) \mid g \in \mathcal{G}\} \subseteq \mathbf{LM}(G) = \{\mathbf{LM}(g') \mid g' \in G\}.$$

Therefore, any two minimal left Gröbner bases of  $N$  have the same set of leading monomials  $\Omega$ .

**Proof** This follows immediately from the definition of a left Gröbner basis, the construction of  $\Omega$  and Proposition 2.2.9.  $\square$

We next introduce the notion of a reduced left Gröbner basis.

**2.2.11. Definition** Let  $N \neq \{0\}$  be a submodule of  $L$  and let  $\mathcal{G}$  be a left gröbner basis of  $N$ . If  $\mathcal{G}$  satisfies the following conditions:

- (1)  $\mathcal{G}$  is a minimal left Gröbner basis;
- (2)  $\mathbf{LC}(g) = 1$  for all  $g \in \mathcal{G}$ ;
- (3) For every  $g \in \mathcal{G}$ ,  $\xi = g - \mathbf{LM}(g)$  is a normal element (mod  $\mathcal{G}$ ), i.e.,  $\xi \in K\text{-span}\mathcal{N}(\mathcal{G})$ ,

then  $\mathcal{G}$  is called a *reduced left Gröbner basis* of  $N$ .

**2.2.12. Proposition** Every nonzero submodule  $N$  of  $L$  has a unique reduced Gröbner basis.

**Proof** Note that if  $\mathcal{G}$  and  $\mathcal{G}'$  are reduced left Gröbner bases for  $N$ , then  $\mathbf{LM}(\mathcal{G}) = \mathbf{LM}(\mathcal{G}')$  by Corollary 2.2.10. If  $\mathbf{LM}(g_i) = \mathbf{LM}(g'_j)$  then  $g_i - g'_j \in N \cap K\text{-span}\mathcal{N}(\mathcal{G}) = N \cap K\text{-span}\mathcal{N}(\mathcal{G}')$ . It follows from Proposition 2.2.7(iv) that  $g_i = g'_j$ . Hence  $\mathcal{G} = \mathcal{G}'$ .  $\square$

The next proposition shows that if a finite left Gröbner basis  $\mathcal{G}$  of  $N$  is given, then a minimal left Gröbner basis, thereby the reduced left Gröbner basis for  $N$  can be obtained in an algorithmic way.

**2.2.13. Proposition** Let  $N \neq \{0\}$  be a submodule of  $L$ , and let  $\mathcal{G} = \{g_1, \dots, g_m\}$  be a finite left Gröbner basis of  $N$ .

(i) The subset  $\mathcal{G}_0 = \{g_i \in \mathcal{G} \mid \mathbf{LM}(g_i) \text{ is minimal in } \mathbf{LM}(\mathcal{G}) \text{ w.r.t. } \prec'_e\}$  of  $\mathcal{G}$  forms a minimal left Gröbner basis of  $N$  (see the definition of  $\prec'_e$  given before Theorem 2.2.6). An algorithm written in pseudo-code is omitted here.

(ii) With  $\mathcal{G}_0$  as in (i) above, we may assume, without loss of generality, that  $\mathcal{G}_0 = \{g_1, \dots, g_s\}$  with  $\mathbf{LC}(g_i) = 1$  for  $1 \leq i \leq s$ . Put  $\mathcal{G}_1 = \{g_2, \dots, g_s\}$  and  $\xi_1 = \overline{g_1}^{\mathcal{G}_1}$ . Then  $\mathbf{LM}(\xi_1) = \mathbf{LM}(g_1)$ . Put  $\mathcal{G}_2 = \{\xi_1, g_3, \dots, g_s\}$  and  $\xi_2 = \overline{g_2}^{\mathcal{G}_2}$ . Then  $\mathbf{LM}(\xi_2) = \mathbf{LM}(g_2)$ . Put  $\mathcal{G}_3 = \{\xi_1, \xi_2, g_4, \dots, g_s\}$  and  $\xi_3 = \overline{g_3}^{\mathcal{G}_3}$ , and so on. The last obtained  $\mathcal{G}_{s+1} = \{\xi_1, \dots, \xi_{s-1}\} \cup \{\xi_s\}$  is then the reduced left Gröbner basis. An algorithm written in pseudocode is omitted here.

**Proof** This can be verified directly, so we leave it as an exercise.  $\square$

### The existence of finite left Gröbner bases

Finally we show that every nonzero submodule  $N$  of the free  $A$ -module  $L = \bigoplus_{i=1}^s Ae_i$  has a finite left Gröbner basis  $\mathcal{G}$  with respect to a given left monomial ordering  $\prec_e$ .

Let  $a^\alpha e_i, a^\beta e_j \in \mathcal{B}(e)$  with  $\alpha = (\alpha_1, \dots, \alpha_n), \beta = (\beta_1, \dots, \beta_n) \in \mathbb{N}^n$ . Recall that  $a^\alpha e_i \prec'_e a^\beta e_j$  if and only if  $i = j$  and  $a^\alpha|_L a^\beta$ , while the latter is defined subject to the property that  $a^\beta = \mathbf{LM}(a^\gamma a^\alpha)$  for some  $a^\gamma \in \mathcal{B}$ .

Thus, by Lemma 2.1.2(ii),  $a^\alpha e_i \prec'_e a^\beta e_j$  is actually equivalent to  $i = j$  and  $\alpha_i \leq \beta_i$  for  $1 \leq i \leq n$ . Also note that the correspondence  $\alpha = (\alpha_1, \dots, \alpha_n) \longleftrightarrow a^\alpha$  gives the bijection between  $\mathbb{N}^n$  and  $\mathcal{B}$ . Combining this observation with Dickson's lemma (Lemma 1.3.1), we are able to reach the following result.

**2.2.14. Theorem** With notations as in Theorem 2.2.6, let  $N \neq \{0\}$  be a submodule of  $L$ ,  $\mathbf{LM}(N) = \{\mathbf{LM}(\xi) \mid \xi \in N\}$ ,  $\Omega = \{a^\alpha e_i \in \mathbf{LM}(N) \mid a^\alpha e_i \text{ is minimal in } \mathbf{LM}(N) \text{ w.r.t. } \prec'_e\}$ , and  $\mathcal{G} = \{g \in N \mid \mathbf{LM}(g) \in \Omega\}$ . Then  $\Omega$  is a finite subset of  $\mathbf{LM}(N)$ , thereby  $\mathcal{G}$  is a finite left Gröbner basis of  $N$ .

**Proof** For  $1 \leq i \leq s$ , put

$$\begin{aligned}\mathcal{B}(e)_i &= \{a^\alpha e_i \mid a^\alpha \in \mathcal{B}\}, \\ \Omega_i &= \Omega \cap \mathcal{B}(e)_i.\end{aligned}$$

Then  $\Omega = \cup_{i=1}^s \Omega_i$ . Applying Dickson's lemma (Lemma 1.3.1) to  $\Omega_i$ , it turns out that each  $\Omega_i$  has a finite subset  $\Omega'_i = \{a^{\alpha(1_i)} e_i, \dots, a^{\alpha(t_i)} e_i\}$  such that if  $a^\beta e_i \in \Omega_i$ , then  $a^{\alpha(j_i)} e_i \mid_L a^\beta e_i$  for some  $a^{\alpha(j_i)} \in \Omega'_i$ . But this implies  $a^{\alpha(j_i)} e_i \prec'_e a^\beta e_i$  in  $\Omega$ . It follows from the definition of  $\Omega$  that  $\Omega = \cup_{i=1}^s \Omega'_i$  is a finite subset of  $\mathbf{LM}(N)$ , and consequently  $\mathcal{G}$  is a finite (minimal) left Gröbner basis of  $N$ .  $\square$

## 2.3. The Noncommutative Buchberger Algorithm

Recall from the theory of Gröbner bases for commutative polynomial algebras ([Bu1], [Bu2], [AL2], [BW], [Fröb], [KR1], [KR2]) that the celebrated Buchberger algorithm depends on Buchberger's criterion which establishes the strategy for computing Gröbner bases of polynomial ideals. Let  $A = K[a_1, \dots, a_n]$  be a solvable polynomial algebra with admissible system  $(\mathcal{B}, \prec)$ , and let  $L = \oplus_{i=1}^s A e_i$  be a free  $A$ -module with left admissible system  $(\mathcal{B}(e), \prec_e)$ . Then by Theorem 2.2.14, every nonzero submodule  $N$  has a finite left Gröbner basis  $\mathcal{G}$ . In this section, by introducing left S-polynomials for elements  $(\xi, \zeta) \in L \times L$ , we show that a noncommutative version of Buchberger's criterion holds true for submodules of  $L$ , and that a noncommutative version of the Buchberger algorithm works effectively for computing finite left Gröbner bases of submodules in  $L$ .

Since the noncommutative version of Buchberger's criterion and the noncommutative version of Buchberger's algorithm for modules over a



solvable polynomial algebra  $A$  (Theorem 2.3.3 and **Algorithm-LGB** presented below) look as if working the same way as in the commutative case by reducing the S-polynomials, at this stage one is asked to pay more attention to compare the argumentation concerning Buchberger's criterion in the commutative case (e.g. [AL2], P.40-42) and that in the noncommutative case, given in [K-RW] and we are going to give respectively, so as to see how the barrier made by the noncommutativity of  $A$  (i.e., the product of two monomials in  $A$  is no longer necessarily a monomial of  $A$ ) can be broken down. Also, at this point we remind that in the proof of ([K-RW], Theorem 3.11), the argumentation was given in the language of abstract rewriting (see [Wik2] for an introduction of this topic), namely, it was shown that the left reduction of left S-polynomials by  $G$  is locally confluent; while in the argumentation we are going to give below, Lemma 2.3.1 will play the key role as the breakthrough point, though our presentation looks quite similar to the most popularly known presentation in the commutative case (e.g. [AL2]).

Let  $\xi, \zeta$  be nonzero elements of  $L$  with  $\mathbf{LM}(\xi) = a^\alpha e_i$ ,  $\mathbf{LM}(\zeta) = a^\beta e_j$ , where  $\alpha = (\alpha_1, \dots, \alpha_n)$ ,  $\beta = (\beta_1, \dots, \beta_n)$ . Put  $\gamma = (\gamma_1, \dots, \gamma_n)$  with  $\gamma_k = \max\{\alpha_k, \beta_k\}$ . The *left S-polynomial* of  $\xi$  and  $\zeta$  is defined as the element

$$S_\ell(\xi, \zeta) = \begin{cases} \frac{1}{\mathbf{LC}(a^{\gamma-\alpha}\xi)} a^{\gamma-\alpha}\xi - \frac{1}{\mathbf{LC}(a^{\gamma-\beta}\zeta)} a^{\gamma-\beta}\zeta, & \text{if } i = j \\ 0, & \text{if } i \neq j. \end{cases}$$

**Observation** If  $S_\ell(\xi, \zeta) \neq 0$  then, with respect to the given  $\prec_e$  on  $\mathcal{B}(e)$  we have  $\mathbf{LM}(S_\ell(\xi, \zeta)) \prec_e a^\gamma e_i$ .

**2.3.1. Lemma** Let  $\xi, \zeta$  be as above such that  $S_\ell(\xi, \zeta) \neq 0$ , and put  $\lambda = \mathbf{LC}(a^{\gamma-\alpha}\xi)$ ,  $\mu = \mathbf{LC}(a^{\gamma-\beta}\zeta)$ . If there exist  $a^{\theta(1)}, a^{\theta(2)} \in \mathcal{B}$  such that  $\mathbf{LC}(a^{\theta(1)}\xi) = \lambda_1$ ,  $\mathbf{LC}(a^{\theta(2)}\zeta) = \mu_1$ ,  $\mathbf{LM}(a^{\theta(1)}\xi) = a^\rho e_i = \mathbf{LM}(a^{\theta(2)}\zeta)$ , where  $\rho = (\rho_1, \dots, \rho_n)$ , then there exists  $a^\delta \in \mathcal{B}$  such that

$$S_\ell(a^{\theta(1)}\xi, a^{\theta(2)}\zeta) = b \left( a^\delta S_\ell(\xi, \zeta) - d a^{\theta(2)}\zeta - f_1 \xi - f_2 \zeta \right),$$

where  $b = \frac{\lambda}{\lambda_{\rho, \alpha} \lambda_1}$ ,  $d = \frac{\lambda_{\rho, \alpha} \lambda_1}{\lambda_{\mu_1}} - \frac{\mu_{\rho, \beta}}{\mu}$ ,  $\mathbf{LM}(a^\delta S_\ell(\xi, \zeta)) \prec_e a^\rho e_i$ ,  $\mathbf{LM}(f_1 \xi) \prec_e a^\rho e_i$ , and  $\mathbf{LM}(f_2 \zeta) \prec_e a^\rho e_i$ ,  $\lambda_{\rho, \alpha}, \mu_{\rho, \beta} \in K^*$ ,  $f_1, f_2 \in A$ , which are given

in terms of

$$\begin{aligned} a^{\rho-\gamma}a^{\gamma-\alpha} &= \lambda_{\rho,\alpha}a^{\theta(1)} + f_1 \text{ with } f_1 \in A \text{ and } \mathbf{LM}(f_1) \prec a^{\theta(1)}; \\ a^{\rho-\gamma}a^{\gamma-\beta} &= \mu_{\rho,\beta}a^{\theta(2)} + f_2 \text{ with } f_2 \in A \text{ and } \mathbf{LM}(f_2) \prec a^{\theta(2)}. \end{aligned}$$

**Proof** By the assumption and Lemma 2.1.2,  $\theta(1)+\alpha = \rho = \theta(2)+\beta$ . Since  $\gamma = (\gamma_1, \dots, \gamma_n)$  with  $\gamma_k = \max\{\alpha_k, \beta_k\}$ , we may put  $\delta = (\delta_1, \dots, \delta_n)$  with  $\delta_i = \rho_i - \gamma_i$ , i.e.,  $\delta = \rho - \gamma$ . Now we see that

$$\begin{aligned} a^\delta S_\ell(\xi, \zeta) &= \frac{1}{\lambda} a^\delta a^{\gamma-\alpha} \xi - \frac{1}{\mu} a^\delta a^{\gamma-\beta} \zeta \\ &= \frac{1}{\lambda} a^{\rho-\gamma} a^{\gamma-\alpha} \xi - \frac{1}{\mu} a^{\rho-\gamma} a^{\gamma-\beta} \zeta \\ &= \left( \frac{\lambda_{\rho,\alpha}}{\lambda} a^{\theta(1)} \xi + f_1 \xi \right) - \left( \frac{\mu_{\rho,\beta}}{\mu} a^{\theta(2)} \zeta + f_2 \zeta \right), \\ &= \frac{\lambda_{\rho,\alpha} \lambda_1}{\lambda} \left( \frac{1}{\lambda_1} a^{\theta(1)} \xi - \frac{1}{\mu} a^{\theta(2)} \zeta \right) \\ &\quad + \left( \frac{\lambda_{\rho,\alpha} \lambda_1}{\lambda \mu_1} - \frac{\mu_{\rho,\beta}}{\mu} \right) a^{\theta(2)} \zeta + f_1 \xi + f_2 \zeta \\ &= \frac{\lambda_{\rho,\alpha} \lambda_1}{\lambda} S_\ell(a^{\theta(1)} \xi, a^{\theta(2)} \zeta) \\ &\quad + \left( \frac{\lambda_{\rho,\alpha} \lambda_1}{\lambda \mu_1} - \frac{\mu_{\rho,\beta}}{\mu} \right) a^{\theta(2)} \zeta + f_1 \xi + f_2 \zeta, \end{aligned}$$

in which  $\mathbf{LM}(a^\delta S_\ell(\xi, \zeta)) \prec_e a^\rho e_i$ ,  $\mathbf{LM}(f_1 \xi) \prec_e a^\rho e_i$ , and  $\mathbf{LM}(f_2 \zeta) \prec_e a^\rho e_i$ .  $\square$

**2.3.2. Lemma** Let  $\xi_1, \dots, \xi_t \in L$  be such that  $\mathbf{LM}(\xi_i) = a^\rho e_q$  for all  $i = 1, \dots, t$ . If, for  $c_1, \dots, c_t \in K^*$ , the element  $\xi = \sum_{i=1}^t c_i \xi_i$  satisfies  $\mathbf{LM}(\xi) \prec_e a^\rho e_q$ , then  $\xi$  can be expressed as a  $K$ -linear combination of the form

$$\xi = d_{12} S_\ell(\xi_1, \xi_2) + d_{23} S_\ell(\xi_2, \xi_3) + \dots + d_{t-1,t} S_\ell(\xi_{t-1}, \xi_t),$$

where  $d_{ij} \in K$ .

**Proof** If we rewrite each  $\xi_i$  as  $\xi_i = \lambda_i a^\rho e_q + \text{lower terms}$ , where  $\lambda_i = \mathbf{LC}(\xi_i)$ , then the assumption yields a cancelation of leading terms which gives rise to  $\sum_{i=1}^s c_i \lambda_i = 0$ . Since  $\mathbf{LM}(\xi_i) = a^\rho e_q = \mathbf{LM}(\xi_j)$ , it follows that  $S_\ell(\xi_i, \xi_j) = \frac{1}{\lambda_i} \xi_i - \frac{1}{\lambda_j} \xi_j$ . Thus

$$\begin{aligned} \xi &= c_1 \xi_1 + \dots + c_s \xi_t \\ &= c_1 \lambda_1 \left( \frac{1}{\lambda_1} \xi_1 \right) + \dots + c_t \lambda_t \left( \frac{1}{\lambda_t} \xi_t \right) \\ &= c_1 \lambda_1 \left( \frac{1}{\lambda_1} \xi_1 - \frac{1}{\lambda_2} \xi_2 \right) + (c_1 \lambda_1 + c_2 \lambda_2) \left( \frac{1}{\lambda_2} \xi_2 - \frac{1}{\lambda_3} \xi_3 \right) \end{aligned}$$

$$\begin{aligned}
& + \cdots + (c_1\lambda_1 + \cdots + c_{t-1}\lambda_{t-1}) \left( \frac{1}{\lambda_{t-1}}\xi_{t-1} - \frac{1}{\lambda_t}\xi_t \right) \\
& + (c_1\lambda_1 + \cdots + c_t\lambda_t) \frac{1}{\lambda_t}\xi_t \\
& = c_1\lambda_1 S_\ell(\xi_1, \xi_2) + (c_1\lambda_1 + c_2\lambda_2) S_\ell(\xi_2, \xi_3) + \cdots \\
& + (c_1\lambda_1 + \cdots + c_{t-1}\lambda_{t-1}) S_\ell(\xi_{t-1}, \xi_t),
\end{aligned}$$

because  $\sum_{i=1}^t c_i \lambda_i = 0$ . This completes the proof.  $\square$

Considering the submodule  $N = \sum_{i=1}^m A\xi_i$  of  $L$  generated by  $\Xi = \{\xi_1, \dots, \xi_m\} \subset L$ , it is clear that  $S_\ell(\xi_i, \xi_j) \in N$  for  $1 \leq i < j \leq m$ .

**2.3.3. Theorem** (Noncommutative version of Buchberger's criterion) Let  $N = \sum_{i=1}^m A\xi_i$  be a submodule of  $L$  generated by the set of nonzero elements  $\Xi = \{\xi_1, \dots, \xi_s\}$ . Then, with respect to the given left monomial ordering  $\prec_e$  on  $\mathcal{B}(e)$ ,  $\Xi$  is a left Gröbner basis of  $N$  if and only if every nonzero  $S_\ell(\xi_i, \xi_j)$  is reduced to 0 (mod  $\Xi$ ), i.e.,  $\overline{S_\ell(\xi_i, \xi_j)}^\Xi = 0$ .

**Proof** Note that  $S_\ell(\xi_i, \xi_j) \in N$ . If  $\Xi$  is a left Gröbner basis of  $N$ , then  $\overline{S_\ell(\xi_i, \xi_j)}^\Xi = 0$  whenever  $S_\ell(\xi_i, \xi_j) \neq 0$ .

Conversely, suppose that  $\overline{S_\ell(\xi_i, \xi_j)}^\Xi = 0$  for every nonzero  $S_\ell(\xi_i, \xi_j)$ . We will show that  $\Xi$  satisfies Proposition 2.2.4(ii), or in other words, that every nonzero element of  $N$  has a left Gröbner representation by  $\Xi$ . For this purpose, we argue by contradiction. Suppose that there were a  $\xi = \sum_{i=1}^s f_i \xi_i \in N$  with nonzero  $f_i \in A$ , such that  $\mathbf{LM}(\xi) \prec_e \mathbf{LM}(f_j \xi_j)$  for some  $j \leq s$ . Let  $\mathbf{LM}(f_j) = a^{\theta(j)}$  and  $\mathbf{LM}(\xi_j) = a^{\alpha(j)} e_{i_j}$ . Comparing the linear expressions of both sides of  $\xi = \sum_{i=1}^s f_i \xi_i$  in terms of base elements in  $\mathcal{B}(e)$ , we may assume, without loss of generality, that for some  $2 \leq t \leq s$ ,

$$\begin{aligned}
a^\rho e_q &= \mathbf{LM}(f_1 \xi_1) = \mathbf{LM}(f_2 \xi_2) = \cdots = \mathbf{LM}(f_t \xi_t) \\
&= \max\{\mathbf{LM}(f_1 \xi_1), \dots, \mathbf{LM}(f_s \xi_s)\},
\end{aligned} \tag{1}$$

and moreover, we assume that the representation  $\xi = \sum_{i=1}^s f_i \xi_i$  is chosen so that  $a^\rho e_q$  is the least one. We now proceed to produce a new representation of  $\xi$  by elements of  $\Xi$  in which the maximal monomial is strictly less than  $a^\rho e_q$ . To this end, we write  $f_i = c_i a^{\theta(i)} + \text{lower terms}$ , where  $c_i = \mathbf{LC}(f_i)$ , and let  $\eta_0 := \sum_{i=1}^t c_i a^{\theta(i)} \xi_i$ ,  $\xi^* = \xi - \eta_0$ . Then  $\mathbf{LM}(\eta_0) \prec_e a^\rho e_q$ ,  $\mathbf{LM}(\xi^*) \prec_e a^\rho e_q$ . With  $\lambda_i = \mathbf{LC}(a^{\theta(i)} \xi_i)$ ,  $1 \leq i \leq t$ , we

may apply Lemma 2.3.2 to  $\eta_0$  so that

$$\begin{aligned} \eta_0 = & d_{12}S_\ell(a^{\theta(1)}\xi_1, a^{\theta(2)}\xi_2) + d_{23}S_\ell(a^{\theta(2)}\xi_2, a^{\theta(3)}\xi_3) + \cdots \\ & + d_{t-1,t}S_\ell(a^{\theta(t-1)}\xi_{t-1}, a^{\theta(t)}\xi_t), \end{aligned} \quad (2)$$

where  $d_{ij} \in K$  and  $S_\ell(a^{\theta(i)}\xi_i, a^{\theta(j)}\xi_j) = \frac{1}{\lambda_i}a^{\theta(i)}\xi_i - \frac{1}{\lambda_j}a^{\theta(j)}\xi_j$ . We next apply Lemma 2.3.1 to each  $S_\ell(a^{\theta(i)}\xi_i, a^{\theta(j)}\xi_j)$  so that

$$S_\ell(a^{\theta(i)}\xi_i, a^{\theta(j)}\xi_j) = b_{ij} \left( a^{\delta(ij)}S_\ell(\xi_i, \xi_j) - z_{ij}a^{\theta(j)}\xi_j - h_i\xi_i - h_j\xi_j \right), \quad (3)$$

where  $b_{ij}, z_{ij} \in K, h_i, h_j \in A, \mathbf{LM}(a^{\delta(ij)}S_\ell(\xi_i, \xi_j)) \prec_e a^\rho e_q, \mathbf{LM}(h_i\xi_i) \prec_e a^\rho e_q$ , and  $\mathbf{LM}(h_j\xi_j) \prec_e a^\rho e_q$ , thereby

$$\mathbf{LM}(S_\ell(a^{\theta(i)}\xi_i, a^{\theta(j)}\xi_j)) = \mathbf{LM}(a^{\theta(j)}\xi_j) = a^\rho e_q \text{ if } b_{ij}, z_{ij} \neq 0. \quad (4)$$

Note that  $j = i + 1 > i$  in (3). After substituting (3) into (2), and  $\eta_0$  into  $\xi$ , we see that  $a^{\theta(1)}\xi_1$  does not appear in the new representation of  $\xi$ . So, if we take the sub-sum  $\eta_1 := \sum_{j=2}^t d_{ij}b_{ij}z_{ij}a^{\theta(j)}\xi_j$  of  $\xi$ , then  $\eta_1$  clearly satisfies  $\mathbf{LM}(\eta_1) \prec_e a^\rho e_q = \mathbf{LM}(a^{\theta(j)}\xi_j), 2 \leq j \leq t$ . Hence we may apply Lemma 2.3.1 and Lemma 2.3.2 to  $\eta_1$  so that we obtain a similar result of (3) which gives rise to a sub-sum  $\eta_2 := \sum_{k=3}^t u_k a^{\theta(k)}\xi_k$  of  $\xi$  with  $u_k \in K$  and  $\mathbf{LM}(\eta_2) \prec_e a^\rho e_q = \mathbf{LM}(a^{\theta(k)}\xi_k), 3 \leq k \leq t$ . Repeating such a procedure for at most  $t - 1$  times and at each time, applying the assumption  $\overline{S_\ell(\xi_i, \xi_j)}^\Xi = 0$  to (3), we eventually obtain a representation of  $\xi$  by elements of  $\Xi$  which yields the desired contradiction and finishes the proof.  $\square$

**2.3.4. Theorem** (The Noncommutative version of Buchberger algorithm) Let  $N = \sum_{i=1}^m A\xi_i$  be a submodule of  $L$  generated by a finite set of nonzero elements  $\Xi = \{\xi_1, \dots, \xi_s\}$ . Then, with respect to a given left monomial ordering  $\prec_e$  on  $\mathcal{B}(e)$ , the algorithm presented below returns a finite left Gröbner basis  $\mathcal{G}$  for  $N$ .

---

#### Algorithm-LGB

---

INPUT:  $\Xi = \{\xi_1, \dots, \xi_m\}$   
 OUTPUT:  $\mathcal{G} = \{g_1, \dots, g_t\}$ , a left Gröbner basis for  $N = \sum_{i=1}^m A\xi_i$   
 INITIALIZATION:  $m' := m, \mathcal{G} := \{g_1 = \xi_1, \dots, g_{m'} = \xi_m\},$   
                    $\mathcal{S} := \{S_\ell(g_i, g_j) \mid g_i, g_j \in \mathcal{G}, i < j\} - \{0\}$   
 BEGIN

```

WHILE  $\mathcal{S} \neq \emptyset$  DO
  Choose any  $S_\ell(g_i, g_j)$  from  $\mathcal{S}$ 
   $\mathcal{S} := \mathcal{S} - \{S_\ell(g_i, g_j)\}$ 
  IF  $\overline{S_\ell(g_i, g_j)}^{\mathcal{G}} = \eta \neq 0$  with  $\mathbf{LM}(\eta) = a^\rho e_k$  THEN
     $m' := m' + 1, g_{m'} := \eta$ 
     $\mathcal{S} := \mathcal{S} \cup \{S_\ell(g_j, g_{m'}) \mid g_j \in \mathcal{G}, \mathbf{LM}(g_j) = a^\nu e_k\} - \{0\}$ 
     $\mathcal{G} := \mathcal{G} \cup \{g_{m'}\},$ 
  END
END
END

```

---

**Proof** We first prove that the algorithm terminates after a finite number of executing the WHILE loop. To this end, let  $\mathcal{G}_{n+1}$  denote the new set obtained after the  $n$ -th turn of executing the WHILE loop, that is,

$$\mathcal{G}_{n+1} = \mathcal{G}_n \cup \left\{ \eta = \overline{S_\ell(g_i, g_j)}^{\mathcal{G}_n} \neq 0 \text{ for some pair } g_i, g_j \in \mathcal{G}_n \right\}, \quad n \in \mathbb{N},$$

where  $\mathcal{G}_0 = \Xi$ , and let

$$\mathcal{G} = \bigcup_{n \in \mathbb{N}} \mathcal{G}_n, \quad \mathbf{LM}(\mathcal{G}) = \{\mathbf{LM}(g) \mid g \in \mathcal{G}\}.$$

By Dickson's lemma (Lemma 1.3.1),  $\mathbf{LM}(\mathcal{G})$  has a finite subset  $U = \{\mathbf{LM}(g_1), \dots, \mathbf{LM}(g_s)\}$ , such that if  $g \in \mathcal{G}$  then  $\mathbf{LM}(g_i) \mid \mathbf{LM}(g)$  for some  $\mathbf{LM}(g_i) \in U$ . Since  $U$  is finite, we may assume that  $U \subset \mathbf{LM}(\mathcal{G}_k)$  for some  $k$ . This shows that  $\overline{S_\ell(g_i, g_j)}^{\mathcal{G}_k} = 0$  for all  $g_i, g_j \in \mathcal{G}_k$ , thereby the algorithm terminates after the  $k$ -th turn of executing the WHILE loop.

Now that the algorithm terminates in a finite number of executions, we may assume that  $\mathcal{G} = \{g_1, \dots, g_t\}$ . Then, since  $\mathcal{G}_0 = \Xi \subset \mathcal{G}$ , the algorithm itself tells us that  $\mathcal{G}$  generates the submodule  $N$  and  $\overline{S_\ell(g_i, g_j)}^{\mathcal{G}} = 0$  for all  $g_i, g_j \in \mathcal{G}$ . It follows from Theorem 2.3.3 that  $\mathcal{G}$  is a left Gröbner basis for  $N$ .  $\square$

One is referred to the up-to-date computer algebra systems SINGULAR [DGPS] for the implementation of **Algorithm-LGB**. Also, nowadays there have been optimized algorithms, such as the signature-based algorithm for computing Gröbner bases in solvable polynomial algebras [SWMZ], which is based on the celebrated F5 algorithm [Fau], may be used to speed-up the computation of left Gröbner bases for modules.

**2.3.5. Corollary** Let  $N = \sum_{i=1}^m A\xi_i$  be a submodule of  $L$  generated by a finite set of nonzero elements  $\Xi = \{\xi_1, \dots, \xi_m\}$ , and let  $\mathcal{G} = \{g_1, \dots, g_t\}$  be a left Gröbner basis of  $N$  produced by running **Algorithm-LGB**. Then the following statements hold true.

- (i) All the properties listed in Proposition 2.2.7 can be recognized in a computational way.
- (ii) There is a  $t \times m$  matrix

$$V_{t \times m} = \begin{pmatrix} h_{11} & \cdots & h_{1m} \\ \cdots & \cdots & \cdots \\ h_{t1} & \cdots & h_{tm} \end{pmatrix}, \quad h_{ij} \in A$$

which is obtained after  $\mathcal{G}$  is computed by running **Algorithm-LGB**, such that

$$\mathcal{G} = \begin{pmatrix} g_1 \\ \vdots \\ g_t \end{pmatrix} = V_{t \times m} \begin{pmatrix} \xi_1 \\ \vdots \\ \xi_m \end{pmatrix}, \quad 1 \leq j \leq s.$$

(Note that for convenience, we formally used the matrix expression to demonstrate an obvious meaning.)

- (iii) If  $\xi \in N$  and  $\xi \neq 0$ , then a representation of  $\xi$  by  $\Xi = \{\xi_1, \dots, \xi_m\}$ , say  $\xi = \sum_{i=1}^m f_i \xi_i$ , can be computed.

**Proof** (i) Now that a left Gröbner basis  $\mathcal{G}$  of  $N$  has been computed, all the properties listed in Proposition 2.2.7 can be recognized by means of the division by  $\mathcal{G}$ .

- (ii) Recall from **Algorithm-LGB** that for each  $g_j \in \mathcal{G}$ , either  $g_j \in \Xi = \{\xi_1, \dots, \xi_m\}$  or  $g_j$  is a newly added member of  $\mathcal{G}$  obtained after a certain pass through the WHILE loop. So, starting with the expressions obtained from the first pass through the WHILE loop:

$$S_\ell(\xi_i, \xi_j) = \sum_k h_k \xi_k + \overline{S_\ell(\xi_i, \xi_j)}^\Xi, \quad h_k \in A, \quad 1 \leq i < j \leq m,$$

and keeping track of the linear combinations that give rise to the new elements of  $\mathcal{G}$ , the algorithm eventually yields the desired matrix  $V_{t \times m}$ .

- (iii) By Proposition 2.2.7(iii), after dividing by  $\mathcal{G}$  the element  $\xi$  has a representation  $\xi = \sum_{j=1}^t s_j g_j$  with  $s_j \in A$  and  $g_j \in \mathcal{G}$ . Now the conclusion (ii) yields the desired representation of  $\xi$  by  $\Xi$ .

### 3. Finite Free Resolutions

Since every solvable polynomial algebra  $A$  is (left and right) Noetherian (Section 3 of Chapter 1), and every nonzero submodule of a free left  $A$ -module has a finite left Gröbner basis which can be produced by running **Algorithm-LGB** (Section 3 of Chapter 2), starting from this chapter we shall successively present some details concerning applications of Gröbner bases in constructing finite free resolutions over an arbitrary solvable polynomial algebra  $A$ , minimal finite  $\mathbb{N}$ -graded free resolutions over an  $\mathbb{N}$ -graded solvable polynomial algebra  $A$  with the degree-0 homogenous part being the ground field  $K$ , and minimal finite  $\mathbb{N}$ -filtered free resolutions over an  $\mathbb{N}$ -filtered solvable polynomial algebra  $A$  (where the  $\mathbb{N}$ -filtration of  $A$  is determined by a positive-degree function).

Let  $A$  be a solvable polynomial algebra as before. The current chapter consists of four sections. In Section 1 we demonstrate, for a finitely generated submodule  $N$  of a free left  $A$ -module  $L$ , how to compute a generating set of the syzygy module  $\text{Syz}(N)$  of  $N$  via computing a left Gröbner basis  $\mathcal{G}$  of  $N$ . In Section 2 we show, in a constructive way, that a noncommutative version of Hilbert's syzygy theorem holds true for  $A$ , and consequently, that a finite free resolution can be algorithmically constructed for every finitely generated  $A$ -module. In Section 3, the noncommutative version of Hilbert's syzygy theorem is applied to highlight two homological properties of  $A$ , that is,  $A$  has finite global homological dimension, and every finitely generated projective  $A$ -module is stably free. Based on Section 1 – Section 3, the final Section 4 is devoted to the

calculation of projective dimension of a finitely generated  $A$ -module  $M$ , and meanwhile, the proposed algorithmic procedure verifies whether  $M$  is a projective module or not.

The main references of this chapter are [AL2], [AF], [Rot], [Li1], [GV], [Lev], [DGPS].

Throughout this chapter, modules are meant left modules over solvable polynomial algebras, and all notions and notations used in previous chapters are maintained.

### 3.1. Computation of Syzygies

Let  $A = K[a_1, \dots, a_n]$  be a solvable polynomial algebra with admissible system  $(\mathcal{B}, \prec)$  in the sense of Definition 1.1.3, where  $\mathcal{B} = \{a^\alpha = a_1^{\alpha_1} \cdots a_n^{\alpha_n} \mid \alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n\}$  is the PBW  $K$ -basis of  $A$  and  $\prec$  is a monomial ordering on  $\mathcal{B}$ . Let  $L_0 = \bigoplus_{i=1}^s A e_i$  be a free left  $A$ -module with  $A$ -basis  $\{e_1, \dots, e_s\}$ , and  $\prec_e$  a left monomial ordering on the  $K$ -basis  $\mathcal{B}(e) = \{a^\alpha e_i \mid a^\alpha \in \mathcal{B}, 1 \leq i \leq s\}$  of  $L_0$ . As in (Section 3 of Chapter 2) we write  $S_\ell(\xi, \zeta)$  for the left S-polynomial of two elements  $\xi, \zeta \in L_0$ , that is, if  $\mathbf{LM}(\xi) = a^\alpha e_i$  with  $\alpha = (\alpha_1, \dots, \alpha_n)$ ,  $\mathbf{LM}(\zeta) = a^\beta e_j$  with  $\beta = (\beta_1, \dots, \beta_n)$ , then

$$S_\ell(\xi, \zeta) = \begin{cases} \frac{1}{\mathbf{LC}(a^{\gamma-\alpha}\xi)} a^{\gamma-\alpha}\xi - \frac{1}{\mathbf{LC}(a^{\gamma-\beta}\zeta)} a^{\gamma-\beta}\zeta, & \text{if } i = j \\ 0, & \text{if } i \neq j \end{cases}$$

where  $\gamma = (\gamma_1, \dots, \gamma_n)$  with  $\gamma_k = \max\{\alpha_k, \beta_k\}$ ; moreover, if  $S_\ell(\xi, \zeta) \neq 0$ , then with respect to the given  $\prec_e$  on  $\mathcal{B}(e)$  we have  $\mathbf{LM}(S_\ell(\xi, \zeta)) \prec_e a^\gamma e_i$ .

Let  $N = \sum_{i=1}^m A \xi_i$  be a submodule of  $L_0$  generated by the set of nonzero elements  $U = \{\xi_1, \dots, \xi_m\}$ , and let  $L_1 = \bigoplus_{i=1}^m A \omega_i$  be the free  $A$ -module with  $A$ -basis  $\{\omega_1, \dots, \omega_m\}$ . Then the syzygy module of  $U$  (or equivalently the syzygy module of  $N$ ), denoted  $\text{Syz}(U)$ , is the submodule of  $L_1$  defined by

$$\text{Syz}(U) = \left\{ \sum_{i=1}^m h_i \omega_i \in L_1 \mid \sum_{i=1}^m h_i \xi_i = 0 \right\}.$$

Our aim of this section is to show that a generating set of the syzygy module  $\text{Syz}(U)$  can be computed by means of a left Gröbner basis of  $N$ . To this end, let  $\mathcal{G} = \{g_1, \dots, g_t\}$  be a left Gröbner basis of  $N$  with respect



to  $\prec_e$ , then every nonzero left S-polynomial  $S_\ell(g_i, g_j)$  has a left Gröbner representation  $S_\ell(g_i, g_j) = \sum_{i=1}^t f_i g_i$  with  $\mathbf{LM}(f_i g_i) \preceq_e \mathbf{LM}(S_\ell(g_i, g_j))$  whenever  $f_i \neq 0$  (note that such a representation is obtained by using the division by  $\mathcal{G}$  during executing the WHILE loop of **Algorithm-LGB** presented in (Chapter 2, Theorem 2.3.4)). Considering the syzygy module  $\text{Syz}(\mathcal{G})$  of  $\mathcal{G}$  in the free  $A$ -module  $L_2 = \oplus_{i=1}^t A\varepsilon_i$  with  $A$ -basis  $\{\varepsilon_1, \dots, \varepsilon_t\}$ , if we put

$$\begin{aligned} s_{ij} &= f_1 \varepsilon_1 + \dots + \left( f_i - \frac{a^{\gamma - \alpha(i)}}{\mathbf{LC}(a^{\gamma - \alpha(i)} \xi_i)} \right) \varepsilon_i + \dots \\ &\quad + \left( f_j + \frac{a^{\gamma - \alpha(j)}}{\mathbf{LC}(a^{\gamma - \alpha(j)} \xi_j)} \right) \varepsilon_j + \dots + f_t \varepsilon_t, \\ \mathcal{S} &= \{s_{ij} \mid 1 \leq i < j \leq t\}, \end{aligned}$$

then it can be shown, actually as in the commutative case (cf. [AL2], Theorem 3.7.3), that  $\mathcal{S}$  generates  $\text{Syz}(\mathcal{G})$  in  $L_2$ . However, by employing the Schreyer ordering  $\prec_{s-\varepsilon}$  on the  $K$ -basis  $\mathcal{B}(\varepsilon) = \{a^\alpha \varepsilon_i \mid a^\alpha \in \mathcal{B}, 1 \leq i \leq m\}$  of  $L_2$  induced by  $\mathcal{G}$  with respect to  $\prec_e$  (see Section 1 of Chapter 2), which is defined subject to the rule: for  $a^\alpha \varepsilon_i, a^\beta \varepsilon_j \in \mathcal{B}(\varepsilon)$ ,

$$a^\alpha \varepsilon_i \prec_{s-\varepsilon} a^\beta \varepsilon_j \Leftrightarrow \begin{cases} \mathbf{LM}(a^\alpha g_i) \prec_e \mathbf{LM}(a^\beta g_j), \\ \text{or} \\ \mathbf{LM}(a^\alpha g_i) = \mathbf{LM}(a^\beta g_j) \text{ and } i < j, \end{cases}$$

there is indeed a much stronger result, namely the noncommutative analogue of Schreyer theorem [Sch] (cf. Theorem 3.7.13 in [AL2] for free modules over commutative polynomial algebras; Theorem 4.8 in [Lev] for free modules over solvable polynomial algebras):

**3.1.1. Theorem (2.3.1)** With respect to the left monomial ordering  $\prec_{s-\varepsilon}$  on  $\mathcal{B}(\varepsilon)$  as defined above, the following statements hold.

- (i) Let  $s_{ij}$  be determined by  $S_\ell(g_i, g_j)$ , where  $i < j$ ,  $\mathbf{LM}(g_i) = a^{\alpha(i)} e_s$  with  $\alpha(i) = (\alpha_{i_1}, \dots, \alpha_{i_n})$ , and  $\mathbf{LM}(g_j) = a^{\alpha(j)} e_s$  with  $\alpha(j) = (\alpha_{j_1}, \dots, \alpha_{j_n})$ . Then  $\mathbf{LM}(s_{ij}) = a^{\gamma - \alpha(j)} \varepsilon_j$ , where  $\gamma = (\gamma_1, \dots, \gamma_n)$  with each  $\gamma_k = \max\{\alpha_{i_k}, \alpha_{j_k}\}$ .
- (ii)  $\mathcal{S}$  is a left Gröbner basis of  $\text{Syz}(\mathcal{G})$ , thereby  $\mathcal{S}$  generates  $\text{Syz}(\mathcal{G})$ .

**Proof** (i) By the definition of  $S_\ell(g_i, g_j)$  we know that  $\mathbf{LM}(a^{\gamma - \alpha(i)} g_i) = a^\gamma e_s = \mathbf{LM}(a^{\gamma - \alpha(j)} g_j)$ . Since  $i < j$ , by the definition of  $\prec_{s-\varepsilon}$  we have

$a^{\gamma-\alpha(i)}\varepsilon_i \prec_{s-\varepsilon} a^{\gamma-\alpha(j)}\varepsilon_j$ . Consequently, it follows from  $\mathbf{LM}(S_\ell(g_i, g_j)) \prec_e \mathbf{LM}(a^{\gamma-\alpha(j)}g_n)$ , the Gröbner representation  $S_\ell(g_i, g_j) = \sum_{i=1}^t f_i g_i$ , and the definition of  $\prec_{s-\varepsilon}$  that  $\mathbf{LM}(s_{ij}) = a^{\gamma-\alpha(j)}\varepsilon_j$ .

(ii) To show that  $\mathcal{S}$  forms a left Gröbner basis for  $\text{Syz}(\mathcal{G})$  with respect to  $\prec_{s-\varepsilon}$ , take any nonzero  $\xi \in \text{Syz}(\mathcal{G})$ . After executing the division algorithm by  $\mathcal{S}$ ,  $\xi$  has an expression  $\xi = \sum_{s_{ij} \in \mathcal{S}} f_{ij}s_{ij} + \eta$ , where  $\mathbf{LM}(f_{ij}s_{ij}) \preceq_{s-\varepsilon} \mathbf{LM}(\xi)$  for each term  $f_{ij}s_{ij}$ , and  $\eta = \bar{\xi}^{\mathcal{S}}$  is the remainder of  $\xi$  on division by  $\mathcal{S}$ . We claim that  $\eta = 0$ , thereby  $\mathcal{S}$  is a left Gröbner basis for  $\text{Syz}(\mathcal{G})$ . Otherwise, assuming the contrary that  $\eta = \sum_{i,j} \lambda_{ij} a^{\theta(i_j)}\varepsilon_j \neq 0$ , where  $\lambda_{ij} \in K^*$  and  $\theta(i_j) = (\theta_{i_{j1}}, \dots, \theta_{i_{jn}}) \in \mathbb{N}^n$ , then  $\mathbf{LM}(\eta) = a^{\theta(l_k)}\varepsilon_k$  for some pair  $(l, k)$  and

$$\mathbf{LM}(s_{ij}) \not\prec a^{\theta(l_k)}\varepsilon_k \text{ for all } s_{ij} \in \mathcal{S}. \quad (1)$$

Let  $s_{ij}$  be determined by  $S_\ell(g_i, g_j)$ , where  $i < j$ ,  $\mathbf{LM}(g_i) = a^{\alpha(i)}e_s$  and  $\mathbf{LM}(g_j) = a^{\alpha(j)}e_s$ . Then by (i) we have  $\mathbf{LM}(s_{ij}) = a^{\gamma-\alpha(j)}\varepsilon_j$ . If  $j = k$  then by (1),

$$a^{\gamma-\alpha(k)} \not\prec a^{\theta(l_k)}. \quad (2)$$

Since  $\mathbf{LM}(\eta) = a^{\theta(l_k)}\varepsilon_k$ , by the definition of  $\prec_{s-\varepsilon}$  we have  $\mathbf{LM}(a^{\theta(i_j)}g_j) \preceq_e \mathbf{LM}(a^{\theta(l_k)}g_k)$ , and for  $(i, j) \neq (l, k)$ ,

$$\mathbf{LM}(a^{\theta(i_j)}g_j) = \mathbf{LM}(a^{\theta(l_k)}g_k) \text{ implies } j < k. \quad (3)$$

Noticing  $\eta = \xi - \sum_{s_{ij} \in \mathcal{S}} f_{ij}s_{ij} \in \text{Syz}(\mathcal{G})$ , we have  $\sum_{i,j} \lambda_{ij} a^{\theta(i_j)}g_j = 0$ . Since  $A$  is a domain, if  $\mathbf{LM}(a^{\theta(i_j)}g_j) \neq \mathbf{LM}(a^{\theta(l_k)}g_k)$  for all  $(i, j) \neq (l, k)$ , then we would have  $\eta = 0$  (note that all  $g_i \neq 0$ ), which is a contradiction. So, by (3) we may assume that  $\mathbf{LM}(a^{\theta(i_j)}g_j) = \mathbf{LM}(a^{\theta(l_k)}g_k)$  for some  $j < k$ . Let  $\mathbf{LM}(g_j) = a^{\alpha(j)}e_s$  and  $\mathbf{LM}(g_k) = a^{\alpha(k)}e_s$ . Then

$$\begin{aligned} \mathbf{LM}(a^{\theta(i_j)}g_j) &= \mathbf{LM}(a^{\theta(i_j)}\mathbf{LM}(g_j)) = \mathbf{LM}(a^{\theta(i_j)}a^{\alpha(j)}e_s) = a^{\theta(i_j)+\alpha(j)}e_s, \\ \mathbf{LM}(a^{\theta(l_k)}g_k) &= \mathbf{LM}(a^{\theta(l_k)}\mathbf{LM}(g_k)) = \mathbf{LM}(a^{\theta(l_k)}a^{\alpha(k)}e_s) = a^{\theta(l_k)+\alpha(k)}e_s, \end{aligned}$$

thereby  $\theta(i_j) + \alpha(j) = \theta(l_k) + \alpha(k)$ . Now, taking  $\gamma = (\gamma_1, \dots, \gamma_n)$  in which  $\gamma_\ell = \max\{\alpha_{j_\ell}, \alpha_{k_\ell}\}$  with respect to  $\alpha(j) = (\alpha_{j_1}, \dots, \alpha_{j_n})$  and  $\alpha(k) = (\alpha_{k_1}, \dots, \alpha_{k_n})$ , it follows that  $\theta(l_k) + \alpha(k) = \rho + \gamma$  for some  $\rho = (\rho_1, \dots, \rho_n)$ . Hence  $\theta(l_k) = \rho + (\gamma - \alpha(k))$ , and this gives rise to  $a^{\theta(l_k)} = \mathbf{LM}(a^\rho a^{\gamma-\alpha(k)})$ , i.e.,  $a^{\gamma-\alpha(k)}|a^{\theta(l_k)}$ , contradicting (2). Therefore, we must have  $\eta = 0$ , as claimed. This completes the proof.  $\square$

To go further, again let  $\mathcal{G} = \{g_1, \dots, g_t\}$  be the left Gröbner basis of  $N$  produced by running **Algorithm-LGB** presented in (Chapter 2, Theorem 2.3.4) with the initial input data  $U = \{\xi_1, \dots, \xi_m\}$ . Using the usual matrix notation for convenience, we have

$$\begin{pmatrix} \xi_1 \\ \vdots \\ \xi_m \end{pmatrix} = U_{m \times t} \begin{pmatrix} g_1 \\ \vdots \\ g_t \end{pmatrix}, \quad \begin{pmatrix} g_1 \\ \vdots \\ g_t \end{pmatrix} = V_{t \times m} \begin{pmatrix} \xi_1 \\ \vdots \\ \xi_m \end{pmatrix},$$

where the  $m \times t$  matrix  $U_{m \times t}$  (with entries in  $A$ ) is obtained by the division by  $\mathcal{G}$ , and the  $t \times m$  matrix  $V_{t \times m}$  (with entries in  $A$ ) is obtained by keeping track of the reductions during executing the WHILE loop of **Algorithm-LGB** (Chapter 2, Corollary 2.3.5). By Theorem 3.1.1, we may write  $\text{Syz}(\mathcal{G}) = \sum_{i=1}^r A\mathcal{S}_i$  with  $\mathcal{S}_1, \dots, \mathcal{S}_r \in L_2 = \oplus_{i=1}^t A\varepsilon_i$ ; and if  $\mathcal{S}_i = \sum_{j=1}^t f_{ij}\varepsilon_j$ , then we write  $\mathcal{S}_i$  as a  $1 \times t$  row matrix, i.e.,  $\mathcal{S}_i = (f_{i1} \dots f_{it})$ , whenever matrix notation is convenient in the according discussion. At this point, we note also that all the  $\mathcal{S}_i$  may be written down one by one during executing the WHILE loop of **Algorithm-LGB** successively. Furthermore, we write  $D_{(1)}, \dots, D_{(m)}$  for the rows of the matrix  $D_{m \times m} = U_{m \times t}V_{t \times m} - E_{m \times m}$  where  $E_{m \times m}$  is the  $m \times m$  identity matrix. The following proposition is a noncommutative analogue of ([AL2], Theorem 3.7.6).

**3.1.2. Theorem** With notation fixed above, the syzygy module  $\text{Syz}(U)$  of  $U = \{\xi_1, \dots, \xi_m\}$  is generated by

$$\{\mathcal{S}_1 V_{t \times m}, \dots, \mathcal{S}_r V_{t \times m}, D_{(1)}, \dots, D_{(m)}\},$$

where each  $1 \times m$  row matrix represents an element of the free  $A$ -module  $L_1 = \oplus_{i=1}^m A\omega_i$ .

**Proof** Since

$$0 = \mathcal{S}_i \begin{pmatrix} g_1 \\ \vdots \\ g_t \end{pmatrix} = (f_{i1} \dots f_{it}) \begin{pmatrix} g_1 \\ \vdots \\ g_t \end{pmatrix} = (f_{i1} \dots f_{it}) V_{t \times m} \begin{pmatrix} \xi_1 \\ \vdots \\ \xi_m \end{pmatrix},$$

we have  $\mathcal{S}_i V_{t \times m} \in \text{Syz}(U)$ ,  $1 \leq i \leq r$ . Moreover, since

$$\begin{aligned}
 D_{m \times m} \begin{pmatrix} \xi_1 \\ \vdots \\ \xi_m \end{pmatrix} &= (U_{m \times t} V_{t \times m} - E_{m \times m}) \begin{pmatrix} \xi_1 \\ \vdots \\ \xi_m \end{pmatrix} \\
 &= U_{m \times t} V_{t \times m} \begin{pmatrix} \xi_1 \\ \vdots \\ \xi_m \end{pmatrix} - \begin{pmatrix} \xi_1 \\ \vdots \\ \xi_m \end{pmatrix} \\
 &= U_{m \times t} \begin{pmatrix} g_1 \\ \vdots \\ g_t \end{pmatrix} - \begin{pmatrix} \xi_1 \\ \vdots \\ \xi_m \end{pmatrix} \\
 &= \begin{pmatrix} \xi_1 \\ \vdots \\ \xi_m \end{pmatrix} - \begin{pmatrix} \xi_1 \\ \vdots \\ \xi_m \end{pmatrix} = 0,
 \end{aligned}$$

we have  $D_{(1)}, \dots, D_{(r)} \in \text{Syz}(U)$ .

On the other hand, if  $H = (h_1 \dots h_m)$  represents the element  $\sum_{i=1}^m h_i \omega_i \in \oplus_{i=1}^m A \omega_i$  such that  $H \begin{pmatrix} \xi_1 \\ \vdots \\ \xi_m \end{pmatrix} = 0$ , then  $0 =$

$HU_{m \times t} \begin{pmatrix} g_1 \\ \vdots \\ g_t \end{pmatrix}$ . This means  $HU_{m \times t} \in \text{Syz}(\mathcal{G})$ . Hence,  $HU_{m \times t} = \sum_{i=1}^r f_i \mathcal{S}_i$  with  $f_i \in A$ , and it follows that  $HU_{m \times t} V_{t \times m} = \sum_{i=1}^r f_i \mathcal{S}_i V_{t \times m}$ . Therefore,

$$\begin{aligned}
 H &= H + HU_{m \times t} V_{t \times m} - HU_{m \times t} V_{t \times m} \\
 &= H(E_m - U_{m \times t} V_{t \times m}) + \sum_{i=1}^r f_i \mathcal{S}_i V_{t \times m} \\
 &= -H D_{m \times m} + \sum_{i=1}^r f_i (\mathcal{S}_i V_{t \times m}).
 \end{aligned}$$

This shows that every element of  $\text{Syz}(U)$  is generated by  $\{\mathcal{S}_1 V_{t \times m}, \dots, \mathcal{S}_r V_{t \times m}, D_{(1)}, \dots, D_{(m)}\}$ , as desired.

### 3.2. Computation of Finite Free Resolutions

Let  $A = K[a_1, \dots, a_n]$  be a solvable polynomial algebra with admissible system  $(\mathcal{B}, \prec)$  in the sense of Definition 1.1.3, where  $\mathcal{B} = \{a^\alpha = a_1^{\alpha_1} \cdots a_n^{\alpha_n} \mid \alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n\}$  is the PBW  $K$ -basis of  $A$  and  $\prec$  is a monomial ordering on  $\mathcal{B}$ . Let  $M$  be a left  $A$ -module. Then a *free resolution* of  $M$  is an *exact sequence* by free  $A$ -modules  $L_i$  and  $A$ -module homomorphisms  $\varphi_i$ :

$$\mathcal{L}_\bullet \quad \cdots \xrightarrow{\varphi_{i+1}} L_i \xrightarrow{\varphi_i} \cdots \xrightarrow{\varphi_2} L_1 \xrightarrow{\varphi_1} L_0 \xrightarrow{\varphi_0} M \longrightarrow 0$$

that is, in the sequence  $\mathcal{L}_\bullet$ ,  $\varphi_0$  is surjective, each  $L_i$  is a free  $A$ -module and  $\text{Ker} \varphi_i = \text{Im} \varphi_{i+1} = \varphi_{i+1}(L_{i+1})$  for all  $i \geq 0$ . By classical homological algebra (e.g. [Rot]), theoretically such a free resolution for  $M$  exists. If  $M$  is a finitely generated  $A$ -module, each  $L_i$  in  $\mathcal{L}_\bullet$  is finitely generated, and  $L_{q+1} = 0$  for some  $q$ , then  $\mathcal{L}_\bullet$  is called a *finite free resolution* of  $M$ .

In this section, we show, in a constructive way, that a noncommutative version of Hilbert's syzygy theorem holds true for  $A$ , i.e., every finitely generated  $A$ -module  $M$  has a finite free resolution, and consequently, an algorithm for computing a finite free resolution of  $M$  is obtained. Our argumentation concerning a noncommutative version of Hilbert's syzygy theorem below is adapted from ([Eis], Corollary 15.11) and ([Lev], Section 4.4).

Let  $L = \oplus_{i=1}^s A e_i$  be a free  $A$ -module with left monomial ordering  $\prec_e$  on its  $K$ -basis  $\mathcal{B}(e)$ , and let  $\mathcal{G} = \{g_1, \dots, g_t\}$  be a left Gröbner basis of the submodule  $N = \sum_{i=1}^t A g_i \subset L$ . Then, after relabeling the members of  $\mathcal{G}$  (if necessary), we may always assume that  $\mathcal{G}$  satisfies

- (\*) if  $i < j$  and  $\mathbf{LM}(g_i) = a^{\alpha(i)} e_k$  with  $\alpha(i) = (\alpha_{i_1}, \dots, \alpha_{i_n})$ ,  $\mathbf{LM}(g_j) = a^{\alpha(j)} e_k$  with  $\alpha(j) = (\alpha_{j_1}, \dots, \alpha_{j_n})$ , then  $a^{\alpha(i)} \preceq_{lex} a^{\alpha(j)}$  under the lexicographic ordering with respect to  $a_n \prec_{lex} a_{n-1} \prec_{lex} \cdots \prec_{lex} a_1$ ,

**3.2.1. Lemma** Given a free  $A$ -module  $L = \oplus_{i=1}^s A e_i$  with left monomial ordering  $\prec_e$  on its  $K$ -basis  $\mathcal{B}(e)$ , let  $\mathcal{G} = \{g_1, \dots, g_t\}$  be a left Gröbner basis of the submodule  $N = \sum_{i=1}^t A g_i \subset L$ . With notation as in Theorem 3.1.1, let  $\prec_{s-\varepsilon}$  be the Schreyer ordering on the  $K$ -basis  $\mathcal{B}(\varepsilon)$  of the free  $A$ -module  $L_1 = \oplus_{i=1}^t A \varepsilon_i$  induced by  $\mathcal{G}$  with respect to  $\prec_e$ . Assume that for some  $r \leq n$ , the generators  $a_1, \dots, a_r$  of  $A$  do not appear in every  $\mathbf{LM}(g_\ell)$ ,

then the generators  $a_1, \dots, a_r, a_{r+1}$  of  $A$  do not appear in  $\mathbf{LM}(s_{ij})$  for every  $s_{ij} \in \mathcal{S}$ .

**Proof** If  $s_{ij} \in \mathcal{S}$  with  $s_{ij} \neq 0$ , then we have  $i < j$  and  $\mathbf{LM}(g_i) = a^{\alpha(i)}e_k$  with  $\alpha(i) = (\alpha_{i_1}, \dots, \alpha_{i_n})$ ,  $\mathbf{LM}(g_j) = a^{\alpha(j)}e_k$  with  $\alpha(j) = (\alpha_{j_1}, \dots, \alpha_{j_n})$ , and it follows from the property (\*) mentioned before the lemma that  $\alpha_{i_\ell} = 0 = \alpha_{j_\ell}$ ,  $1 \leq \ell \leq r$ ,  $\alpha_{i_{r+1}} \leq \alpha_{j_{r+1}}$ . By Theorem 3.1.1,  $\mathbf{LM}(s_{ij}) = a^{\gamma - \alpha(j)}\varepsilon_j$  where  $\gamma = (\gamma_1, \dots, \gamma_n)$  with  $\gamma_k = \max\{\alpha_{i_k}, \alpha_{j_k}\}$ , in particular,  $\gamma_{r+1} = \alpha_{j_{r+1}}$ . This shows that the generators  $a_1, \dots, a_r, a_{r+1}$  of  $A$  do not appear in  $\mathbf{LM}(s_{ij})$ .  $\square$

Let  $M = \sum_{i=1}^s Av_i$  be a finitely generated left  $A$ -module with generating set  $\{v_1, \dots, v_s\}$ . Then since  $A$  is Noetherian, if we consider the free  $A$ -module  $L_0 = \bigoplus_{i=1}^s Ae_i$  and the presentation  $M \cong L_0/N_0$  of  $M$  by a submodule  $N_0$  of  $L_0$ , then  $N_0$  is a finitely generated submodule of  $L_0$ .

**3.2.2. Theorem** (noncommutative version of Hilbert's syzygy theorem) Let  $A = K[a_1, \dots, a_n]$  be a solvable polynomial algebra with admissible system  $(\mathcal{B}, \prec)$ . Then every finitely generated left  $A$ -module  $M$  has a finite free resolution

$$0 \longrightarrow L_q \xrightarrow{\varphi_q} L_{q-1} \xrightarrow{\varphi_{q-1}} \dots \xrightarrow{\varphi_1} L_0 \xrightarrow{\varphi_0} M \longrightarrow 0$$

such that  $q \leq n$ .

**Proof** As remarked above we may assume  $M = L_0/N_0$ , where  $L_0 = \bigoplus_{i=1}^s Ae_i$  is a free  $A$ -module with  $A$ -basis  $\{e_1, \dots, e_s\}$ , and  $N_0$  is a finitely generated submodule of  $L_0$ . Then  $\varphi_0$  is given by the canonical  $A$ -module homomorphism  $L_0 \xrightarrow{\varphi_0} M$ . Fixing a left monomial ordering  $\prec_e$  on the  $K$ -basis  $\mathcal{B}(e)$  of  $L_0$ , let  $\mathcal{G}_0 = \{g_1, \dots, g_{s_1}\}$  be a left Gröbner basis of  $N_0$  as specified before Lemma 3.2.1, such that for some  $r \leq n$ , the generators  $a_1, \dots, a_r$  of  $A$  do not appear in every  $\mathbf{LM}(g_\ell)$ ,  $1 \leq \ell \leq s_1$ . Let  $\prec_{s-\varepsilon}$  be the Schreyer ordering on the  $K$ -basis  $\mathcal{B}(\varepsilon)$  of the free  $A$ -module  $L_1 = \bigoplus_{i=1}^{s_1} A\varepsilon_i$  induced by  $\mathcal{G}_0$  with respect to  $\prec_e$ . Then by Lemma 3.2.1, the generators  $a_1, \dots, a_r, a_{r+1}$  do not appear in  $\mathbf{LM}(s_{ij})$  for every  $s_{ij} \in \mathcal{S}$ , where  $\mathcal{S}$  is the left Gröbner basis of  $N_1 = \text{Syz}(\mathcal{G}_0) \subset L_1$  obtained in Theorem 3.1.1. Defining  $\varphi_1: L_1 \rightarrow L_0$  by  $\varphi_1(\varepsilon_i) = g_i$ ,  $1 \leq i \leq s_1$ , and working with  $N_1$  in place of  $N_0$  and so on, we then reach an exact sequence

$$0 \rightarrow N_{n-r} \longrightarrow L_{n-r} \xrightarrow{\varphi_{n-r}} L_{n-r-1} \xrightarrow{\varphi_{n-r-1}} \dots \xrightarrow{\varphi_2} L_1 \xrightarrow{\varphi_1} L_0 \xrightarrow{\varphi_0} M \rightarrow 0$$

where every  $L_i$  is a free  $A$ -module of finite  $A$ -basis and  $N_{n-r} = \text{Ker}\varphi_{n-r}$  which has a left Gröbner basis  $\mathcal{G}_{n-r} = \{g'_1, \dots, g'_d\}$  in  $L_{n-r}$ , such that all the generators  $a_1, \dots, a_n$  of  $A$  do not appear in every  $\mathbf{LM}(g'_j)$ ,  $1 \leq j \leq d$ . If  $L_{n-r} = \bigoplus_{i=1}^t A\omega_i$ , this shows that  $\mathbf{LM}(g'_j) = \omega_{k_j}$ ,  $1 \leq j \leq d$ . Without loss of generality, we may assume that all the  $\omega_{k_j}$  are distinct (for instance,  $\mathcal{G}_{n-r}$  is minimal), and that  $\mathbf{LM}(g'_j) = \omega_j$ ,  $1 \leq j \leq d$ . Thus, since  $\mathcal{G}_{n-r}$  is a left Gröbner basis, it follows from (Chapter 2, Proposition 2.2.7(iv)) that in this case  $\text{Ker}\varphi_{n-r-1} \cong L_{n-r}/N_{n-r} \cong \bigoplus_{i=d+1}^t A\omega_i$ . Now, with  $L_{n-r}$  replaced by  $\text{Ker}\varphi_{n-r-1}$ , the desired (length  $\leq n$ ) free resolution of  $M$  is obtained.  $\square$

Combining the results of the last section, we are ready to have an algorithmic procedure for constructing a finite free resolution.

**3.2.3. Corollary** Let  $M = L_0/N_0$  be a finitely generated  $A$ -module as in Theorem 3.2.2, and let  $\mathcal{G}_0 = \{g_1, \dots, g_t\}$  be a left Gröbner basis of the submodule  $N_0 = \sum_{i=1}^t Ag_i$  with respect to a left monomial ordering  $\prec_e$  on  $L_0$ . Then, starting with the exact sequence

$$0 \rightarrow N_0 \xrightarrow{\iota} L_0 \xrightarrow{\varphi_0} M \rightarrow 0,$$

where  $\varphi_0$  is the canonical homomorphism, and  $\iota$  is the inclusion map, the following algorithm returns a finite free resolution of  $M$ , which is of length  $q \leq n$ .

---

**Algorithm-FRES**

---

INPUT  $L_0 = \bigoplus_{i=1}^s Ae_i$ ,  $\prec_e$ ,  $\mathcal{G}_0 = \{g_1, \dots, g_t\}$ ,  $L_0 \xrightarrow{\varphi_0} M \rightarrow 0$

OUTPUT  $\mathcal{L}_\bullet$   $0 \rightarrow L_q \xrightarrow{\varphi_q} L_{q-1} \xrightarrow{\varphi_{q-1}} \dots \xrightarrow{\varphi_2} L_1 \xrightarrow{\varphi_1} L_0 \xrightarrow{\varphi_0} M \rightarrow 0$   
a finite free resolution of  $M$

INITIALIZATION  $i := 0$ ,  $\prec := \prec_e$

LOOP

IF all the generators  $a_1, \dots, a_n$  of  $A$  do not appear in  $\mathbf{LM}(g_j)$  for every  $g_j \in \mathcal{G}_i$ , THEN

by the proof of Theorem 3.2.2, there is some  $d$  such that

$$\text{Ker}\varphi_i \cong L_i / \sum_{j=1}^t Ag_j \cong \bigoplus_{i=1}^d Ae_i$$

$$\mathcal{L}_\bullet := (0 \rightarrow \text{Ker}\varphi_i \xrightarrow{\iota} L_i \xrightarrow{\varphi_i} \dots \xrightarrow{\varphi_2} L_1 \xrightarrow{\varphi_1} L_0 \xrightarrow{\varphi_0} M \rightarrow 0)$$

ELSE

$$i := i + 1, L_i := \bigoplus_{j=1}^t Ae_j, \varphi_i := (L_i \rightarrow L_{i-1} \text{ with } \varphi(e_j) = g_j, 1 \leq j \leq t)$$

run **Algorithm-DIV-L** (with respect to  $\prec$ ) to compute a left Gröbner representation of each  $S_\ell(g_k, g_\ell) \neq 0$  by  $\mathcal{G}_{i-1}$  in  $L_{i-1}$ ,  $1 \leq k < \ell \leq t$ , so that a left Gröbner basis  $\mathcal{S} = \{\mathcal{S}_1, \dots, \mathcal{S}_r\}$  of  $\text{Syz}(\mathcal{G}) \subset L_i$  is obtained under the Schreyer ordering  $\prec_{s-e}$  on  $L_i$  induced by  $\mathcal{G}_{i-1}$  with respect to  $\prec$  (Theorem 3.1.1)

$\prec := \prec_{s-e}$

$\mathcal{G}_i := \{g_1, \dots, g_t\}$  with  $g_j = \mathcal{S}_j \neq 0$  for  $1 \leq j \leq t \leq r$

END

UNTIL all the generators  $a_1, \dots, a_n$  of  $A$  do not appear in  $\mathbf{LM}(g_j)$  for every  $g_j \in \mathcal{G}_i$

END

**Proof** By the definition of a free resolution, the sequence  $\mathcal{L}_\bullet$  returned by the algorithm is clearly the desired one for  $M$ . The fact that the **Algorithm-FRES** terminates and returns a sequence  $\mathcal{L}_\bullet$  of finite length  $q \leq n$  is due to Theorem 3.2.2 (or more precisely its proof)

### 3.3. Global Dimension and Stability

In this section, the foregoing Theorem 3.2.2 is applied to highlight that every solvable polynomial algebra  $A$  has finite global homological dimension, and that every finitely generated projective  $A$ -module is stably free.

Let  $A = K[a_1, \dots, a_n]$  be a solvable polynomial algebra with admissible system  $(\mathcal{B}, \prec)$  in the sense of Definition 1.1.3, where  $\mathcal{B} = \{a^\alpha = a_1^{\alpha_1} \cdots a_n^{\alpha_n} \mid \alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n\}$  is the PBW  $K$ -basis of  $A$  and  $\prec$  is a monomial ordering on  $\mathcal{B}$ . Recall from classical homological algebra (e.g. [Rot]) that a left  $A$ -module  $P$  is said to be *projective* if  $M \xrightarrow{\beta} N \rightarrow 0$  is an exact sequence of  $A$ -modules and  $P \xrightarrow{\alpha} N$  is an  $A$ -module homomorphism, then there exists an  $A$ -module homomorphism  $P \xrightarrow{\gamma} M$  such that  $\beta \circ \gamma = \alpha$ , or in other words, the following diagram commutes:

$$\begin{array}{ccccc}
 & & P & & \\
 & \nearrow \gamma & \downarrow \alpha & & \beta \circ \gamma = \alpha \\
 M & \xrightarrow{\beta} & N & \rightarrow & 0
 \end{array}$$



Thus, by the universal property of a free module, any free  $A$ -module is projective and, an  $A$ -module  $P$  is projective if and only if there is a free  $A$ -module  $L$  such that  $P$  is isomorphic to a direct summand of  $L$ , i.e.,  $L \cong P \oplus L_1$ .

Let  $M$  be a left  $A$ -module. Then a *projective resolution* of  $M$  is an *exact sequence* by projective  $A$ -modules  $P_i$  and  $A$ -module homomorphisms  $\varphi_i$ :

$$\mathcal{P}_\bullet \quad \cdots \xrightarrow{\varphi_{i+1}} P_i \xrightarrow{\varphi_i} \cdots \xrightarrow{\varphi_2} P_1 \xrightarrow{\varphi_1} P_0 \xrightarrow{\varphi_0} M \longrightarrow 0$$

that is, in the sequence  $\mathcal{P}_\bullet$ ,  $\varphi_0$  is surjective, each  $P_i$  is a projective  $A$ -module and  $\text{Ker} \varphi_i = \text{Im} \varphi_{i+1} = \varphi_{i+1}(P_{i+1})$  for all  $i \geq 0$ . Clearly, any free resolution of  $M$  is a projective resolution of  $M$ . The *projective dimension* of  $M$ , denoted  $\text{p.dim}_A M$ , is defined to be the shortest length  $q$  of a projective resolution or  $\infty$  if no finite projective resolution exists. The long Schanuel lemma (cf. [Rot]) shows that any projective resolution of  $M$  can be terminated at this length. The *left global dimension* of  $A$  is then defined to be

$$\sup\{\text{p.dim}_A M \mid M \text{ any left } A\text{-module}\}.$$

In terms of right  $A$ -modules, the right global dimension of  $A$  is defined in a similar way. It follows from classical homological algebra (e.g. [Rot]) that the left (right) global dimension is determined by the projective dimension of cyclic modules; moreover, for a (left and right) Noetherian ring  $A$ , the left global dimension of  $A$  is equal to the right global dimension of  $A$ , and this common number is called the *global dimension of  $A$* , denoted  $\text{gl.dim } A$ .

**3.3.1. Theorem** Let  $A = K[a_1, \dots, a_n]$  be a solvable polynomial algebra. Then any  $A$ -module  $M$  has projective dimension  $\text{p.dim}_A M \leq n$ , thereby  $A$  has global dimension  $\text{gl.dim } A \leq n$ .

**Proof** Noticing the classical results on global dimension reviewed above, this follows from the fact that  $A$  is (left and right) Noetherian (Corollary 1.3.3) and the noncommutative version of Hilbert's syzygy theorem (Theorem 3.2.2).  $\square$

Furthermore, since  $A$  is Noetherian,  $A$  has *IBN* (invariant basis number), i.e., for every free  $A$ -module  $L$ , every two  $A$ -bases of  $L$  have the same cardinal (cf. [Rot], Chapter 3). In this case, the *rank* of  $L$ , denoted

$\text{rank}_A L$ , is well defined as the cardinal of some  $A$ -basis of  $L$ , thereby if  $L_1, L_2$  are free  $A$ -modules, then  $L_1 \cong L_2$  if and only if  $\text{rank}_A L_1 = \text{rank}_A L_2$ . With this well-defined rank for free modules, the stably free modules are then well defined, that is, an  $A$ -module  $P$  is said to be *stably free of rank  $t$*  if  $P \oplus L_1 \cong L_2$ , where  $L_1$  is a free module of  $\text{rank}_A L_1 = s$  and  $L_2$  is a free module of  $\text{rank}_A L_2 = s + t$ . Obviously, a stably free module is necessarily finitely generated and projective. It follows from the literature (e.g., [Rot], Chapter 4; [MR], Chapter 11) that stably free  $A$ -modules can be characterized by finite free resolutions.

**3.3.2. Proposition** A finitely generated projective  $A$ -module  $P$  is stably free if and only if  $P$  has a finite free resolution. Furthermore, if

$$\mathcal{L}_\bullet \quad 0 \rightarrow L_q \xrightarrow{\varphi_q} L_{q-1} \xrightarrow{\varphi_{q-1}} \cdots \xrightarrow{\varphi_2} L_1 \xrightarrow{\varphi_1} L_0 \xrightarrow{\varphi_0} P \rightarrow 0$$

is a finite free resolution of  $P$ , then  $\text{rank}_A P = \sum_{i=0}^q (-1)^i \text{rank}_A L_i$ . □

**3.3.3. Theorem** Let  $A = K[a_1, \dots, a_n]$  be a solvable polynomial algebra. Then every finitely generated projective  $A$ -module  $P$  is stably free, and moreover,  $\text{rank}_A P$  is computable via constructing a finite free resolution by running **Algorithm-FRES** given in Corollary 3.2.3.

**Proof** This follows from the noncommutative version of Hilbert's syzygy theorem (Theorem 3.2.2), Theorem 3.2.3, and Proposition 3.3.2 above.

## 3.4. Calculation of $\text{p.dim}_A M$

Let  $A = K[a_1, \dots, a_n]$  be a solvable polynomial algebra with admissible system  $(\mathcal{B}, \prec)$  in the sense of Definition 1.1.3, where  $\mathcal{B} = \{a^\alpha = a_1^{\alpha_1} \cdots a_n^{\alpha_n} \mid \alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n\}$  is the PBW  $K$ -basis of  $A$  and  $\prec$  is a monomial ordering on  $\mathcal{B}$ . Equipped with the previously developed theory and techniques, in this section we establish an algorithmic procedure which calculates the projective dimension of a finitely generated  $A$ -module  $M$  and, at the same time, verifies whether  $M$  is projective or not. The strategy used in our text was proposed by Gago-Vargas in [GV], though the algebras considered in [GV] are restricted to Weyl algebras.

Let  $L \xrightarrow{\varphi} M \rightarrow 0$  be an  $A$ -module epimorphism, and  $K = \text{Ker} \varphi$ . Con-

sider the short exact sequence  $0 \rightarrow K \xrightarrow{\iota} L \xrightarrow{\varphi} M \rightarrow 0$ , where  $\iota$  is the inclusion map. Suppose that there exists an  $A$ -module homomorphism  $M \xrightarrow{\bar{\varphi}} L$  such that  $\varphi \circ \bar{\varphi} = 1_M$ , where  $1_M$  is the identity map of  $M$  to  $M$ . Then, every  $\xi \in L$  has a representation as  $\xi = (\xi - \bar{\varphi}(\varphi(\xi))) + \bar{\varphi}(\varphi(\xi))$  with  $\bar{\varphi}(\varphi(\xi)) \in \text{Im } \bar{\varphi}$  and  $\xi - \bar{\varphi}(\varphi(\xi)) \in K$ . Since it is clear that  $K \cap \text{Im } \bar{\varphi} = \{0\}$  and  $\varphi \circ \bar{\varphi} = 1_M$  implies that  $\bar{\varphi}$  is a monomorphism, it turns out that

$$L = K \oplus \text{Im } \bar{\varphi} = K \oplus \bar{\varphi}(M) \cong K \oplus M. \quad (1)$$

This preliminary enables us to prove the following

**3.4.1. Proposition** Let  $0 \rightarrow L_1 \xrightarrow{\varphi_1} L_0 \xrightarrow{\varphi_0} M \rightarrow 0$  be an exact sequence of  $A$ -modules in which  $L_0, L_1$  are free  $A$ -modules. Then  $M$  is projective if and only if there exists an  $A$ -module homomorphism  $L_0 \xrightarrow{\bar{\varphi}_1} L_1$  such that  $\bar{\varphi}_1 \circ \varphi_1 = 1_{L_1}$ , where  $1_{L_1}$  is the identity map of  $L_1$  to  $L_1$ .

**Proof** Suppose that  $M$  is projective. Then there exists an  $A$ -module homomorphism  $M \xrightarrow{\bar{\varphi}_0} L_0$  such that  $\varphi_0 \circ \bar{\varphi}_0 = 1_M$ , where  $1_M$  is the identity map of  $M$  to  $M$ . Hence, by the formula (1) above we have  $L_0 = \varphi_1(L_1) \oplus \bar{\varphi}_0(M)$ . It follows that if we define  $L_0 \xrightarrow{\bar{\varphi}_1} L_1$  by

$$\bar{\varphi}_1(\varphi_1(\xi_1) + \bar{\varphi}_0(m)) = \xi_1, \quad \xi_1 \in L_1, m \in M,$$

then since  $\varphi_1$  is injective, it is easy to see that  $\bar{\varphi}_1$  is an  $A$ -module homomorphism satisfying  $\bar{\varphi}_1 \circ \varphi_1 = 1_{L_1}$ .

Conversely, suppose that there exists an  $A$ -module homomorphism  $L_0 \xrightarrow{\bar{\varphi}_1} L_1$  such that  $\bar{\varphi}_1 \circ \varphi_1 = 1_{L_1}$ . Then the sequence

$$0 \rightarrow K = \text{Ker } \bar{\varphi}_1 \xrightarrow{\iota} L_0 \xrightarrow{\bar{\varphi}_1} L_1 \rightarrow 0$$

is exact. Since  $L_1$  is free (hence projective), it follows from the formula (1) above that  $L_0 = K \oplus \varphi_1(L_1)$ , thereby  $K \cong L_0 / \varphi_1(L_1) = L_0 / \text{Ker } \varphi_0 \cong M$ . Note that as a direct summand of the free module  $L_0$ ,  $K$  is projective. Hence  $M$  is projective, as desired.  $\square$

Let  $L_0 = \bigoplus_{j=1}^s A e_j$ ,  $L_1 = \bigoplus_{i=1}^t A \varepsilon_i$  be free left  $A$ -modules of rank  $s$  and  $t$  respectively, and let  $L_1 \xrightarrow{\varphi_1} L_0$  be an  $A$ -module homomorphism with  $\varphi_1(\varepsilon_i) = \sum_{j=1}^s f_{ij} e_j$ ,  $1 \leq i \leq t$ . Then  $\varphi_1$  is uniquely determined by

the  $t \times s$  matrix

$$Q_{\varphi_1} = \begin{pmatrix} f_{11} & f_{12} & \cdots & f_{1s} \\ f_{21} & f_{22} & \cdots & f_{2s} \\ \vdots & \vdots & \cdots & \vdots \\ f_{t1} & f_{t2} & \cdots & f_{ts} \end{pmatrix},$$

that is, if  $\xi = \sum_{i=1}^t f_i \varepsilon_i \in L_1$ , then,  $\varphi_1(\xi)$  is given by left multiplication by matrices:

$$\varphi_1(\xi) = \sum_{i=1}^t f_i \varphi_1(\varepsilon_i) = (f_1, \dots, f_t) Q_{\varphi_1} \begin{pmatrix} e_1 \\ \vdots \\ e_s \end{pmatrix}.$$

The  $t \times s$  matrix  $Q_{\varphi_1}$  is usually referred to as the *matrix of  $\varphi_1$* . Thus, in the language of matrices, Proposition 3.4.1 can be restated as follows.

**3.4.2. Proposition** Let the short exact sequence  $0 \rightarrow L_1 \xrightarrow{\varphi_1} L_0 \xrightarrow{\varphi_0} M \rightarrow 0$  be as in Proposition 3.4.1, and let  $Q_{\varphi_1}$  be the matrix of  $\varphi_1$ . Then  $M$  is projective if and only if the  $t \times s$  matrix  $Q_{\varphi_1}$  is right invertible, i.e., there is an  $s \times t$  matrix  $Q_{\overline{\varphi}_1}$  with entries in  $A$ , such that  $Q_{\varphi_1} \cdot Q_{\overline{\varphi}_1} = E_{t \times t}$ , where the latter is the  $t \times t$  identity matrix with the  $(i, i)$ -entry 1  $\in A$ .  $\square$

Furthermore, let the  $t \times s$  matrix  $Q_{\varphi_1}$  of  $\varphi_1$  be as above, and write  $Q_{\varphi_1}^j$  for the  $j$ -th column of  $Q_{\varphi_1}$ ,  $1 \leq j \leq s$ . Considering the free *right*  $A$ -module  $\overline{L}_1 = \oplus_{i=1}^t \varepsilon_i A$  of rank  $t$ , let  $N = \sum_{j=1}^s \xi_j A \subseteq \overline{L}_1$  be the  $A$ -submodule generated by  $\xi_j = (\varepsilon_1, \dots, \varepsilon_t) Q_{\varphi_1}^j = \sum_{i=1}^t \varepsilon_i f_{ij}$ ,  $1 \leq j \leq s$ . Then Proposition 3.4.2 tells us that the left  $A$ -module  $M$  is projective if and only if  $N = \overline{L}_1$ . Since  $A$  has also a *right Gröbner basis theory* for *right modules*, it follows that the following proposition holds true.

**3.4.3. Proposition** With notation as above, let  $\mathcal{G}$  be a right Gröbner basis of the right  $A$ -submodule  $N = \sum_{j=1}^s \xi_j A \subseteq \overline{L}_1 = \oplus_{i=1}^t \varepsilon_i A$ . Then, the left  $A$ -module  $M$  is projective if and only if  $\varepsilon_i \in \mathcal{G}$  for  $1 \leq i \leq t$ . If it is the case, then the right inverse  $Q_{\overline{\varphi}_1}$  of  $Q_{\varphi_1}$  is given by the  $s \times t$  matrix  $V_{s \times t}$  such that

$$(\varepsilon_1, \dots, \varepsilon_t) = (\xi_1, \dots, \xi_s) V_{s \times t},$$

which can be computed via using (Chapter 2, Corollary 2.3.5(ii)).

**Proof** Note that  $N = \overline{L}_1$  if and only if all the  $\varepsilon_i \in N$ ,  $1 \leq i \leq t$ . If  $\mathcal{G}$  is a right Gröbner basis of  $N$ , then by (Chapter 2, Proposition 2.2.7),  $N = \overline{L}_1$  if and only if all the  $\varepsilon_i \in \mathcal{G}$ ,  $1 \leq i \leq t$ . Since  $N = \sum_{j=1}^s \xi_j A$ , it follows from (Chapter 2, Corollary 2.3.5(ii)) that the specified  $s \times t$  matrix  $V_{s \times t}$  can be computed. Also since  $(\xi_1, \dots, \xi_s) = (\varepsilon_1, \dots, \varepsilon_t) Q_{\varphi_1}$ , it turns out that

$$\begin{aligned} (\varepsilon_1, \dots, \varepsilon_t) &= (\xi_1, \dots, \xi_s) V_{s \times t} \\ &= (\varepsilon_1, \dots, \varepsilon_t) Q_{\varphi_1} V_{s \times t}. \end{aligned}$$

This shows that  $V_{s \times t} = Q_{\overline{\varphi}_1}$ , as desired.  $\square$

Now, let  $M$  be a finitely generated  $A$ -module and

$$\mathcal{L}_{\bullet} \quad 0 \rightarrow L_q \xrightarrow{\varphi_q} L_{q-1} \xrightarrow{\varphi_{q-1}} \dots \xrightarrow{\varphi_2} L_1 \xrightarrow{\varphi_1} L_0 \xrightarrow{\varphi_0} M \rightarrow 0$$

a finite free resolution of  $M$  by free modules of finite rank. Suppose that  $\text{Im} \varphi_i$  is projective for some  $i \geq 0$ . Then, by the foregoing discussion, it is not difficult to derive inductively that  $\text{Im} \varphi_{i+k}$  is projective for every  $k = 1, 2, \dots, q - i$ . In particular,  $\text{Im} \varphi_{q-1}$  is projective. By the definition of  $\text{p.dim}_A M$  and the basic property we recalled before Theorem 3.3.1, The next proposition is clear.

**3.4.4. Proposition** With notation as above,  $\text{p.dim}_A M = q$  if and only if  $\text{Im} \varphi_{q-1}$  is not projective if and only if the matrix  $Q_{\varphi_q}$  of  $\varphi_q$  is not right invertible, where the invertibility of  $Q_{\varphi_q}$  can be recognized in a computational way via using Proposition 3.4.3.  $\square$

Suppose that  $\text{Im} \varphi_{q-1}$  is projective. It follows from Proposition 3.4.1 (or its proof) that

$$L_{q-1} = \varphi_q(L_q) \oplus \text{Ker } \overline{\varphi}_q \xrightarrow{\psi} L_q \oplus \text{Im } \varphi_{q-1}, \quad (2)$$

and the latter isomorphism  $\psi$  can be computed, where  $\overline{\varphi}_q$  is the  $A$ -module homomorphism  $L_{q-1} \xrightarrow{\overline{\varphi}_q} L_q$  such that  $\overline{\varphi}_q \circ \varphi_q = 1_{L_q}$ . The formula (2) above enables us to construct another finite free resolution of  $M$

$$0 \rightarrow L_q \xrightarrow{\varphi'_q} L_q \oplus L_{q-1} \xrightarrow{\varphi'_{q-1}} L_q \oplus L_{q-2} \xrightarrow{\varphi'_{q-2}} L_{q-3} \xrightarrow{\varphi_{q-3}} \dots \xrightarrow{\varphi_1} L_0 \xrightarrow{\varphi_0} M \rightarrow 0$$

in which

$$\varphi'_q(\xi_q) = \varphi_q(\xi_q) \text{ for all } \xi_q \in L_q,$$

$$\varphi'_{q-1}(\xi_q + \xi_{q-1}) = \xi_q + \varphi_{q-1}(\xi_{q-1}) \text{ for all } \xi_q + \xi_{q-1} \in L_q \oplus L_{q-1},$$

$$\varphi'_{q-2}(\xi_q + \xi_{q-2}) = \varphi_{q-2}(\xi_{q-2}) \text{ for all } \xi_q + \xi_{q-2} \in L_q \oplus L_{q-2}.$$

By the exactness of free resolution and the formula (2) above, we then have

$$L_{q-1} \xrightarrow{\psi} L_q \oplus \varphi_{q-1}(L_{q-1}) = \text{Im} \varphi'_{q-1} = \text{Ker} \varphi'_{q-2},$$

thereby  $M$  has the following finite free resolution

$$0 \rightarrow L_{q-1} \xrightarrow{\psi} L_q \oplus L_{q-2} \xrightarrow{\varphi'_{q-2}} L_{q-3} \xrightarrow{\varphi_{q-3}} \dots \xrightarrow{\varphi_1} L_0 \xrightarrow{\varphi_0} M \rightarrow 0$$

in which the homomorphism  $\psi$  can be computed. By Proposition 3.4.4, after the projectiveness of  $\text{Im} \varphi_{q-2}$  is checked we can either have  $\text{p.dim}_A M = q - 1$ , or repeat the above procedure again in order to get a finite free resolution of  $M$  which is of length  $q - 2$ . It is clear that after a finite number of repetitions of the same procedure we will eventually have  $\text{p.dim}_A M = m$  with  $m \leq q$ . Obviously, if  $m = 0$ , then  $M$  is projective.

Summing up, we have reached the following

**3.4.5. Theorem** Let  $M$  be a finitely generated  $A$ -module, and let

$$\mathcal{L}_\bullet: 0 \rightarrow L_q \xrightarrow{\varphi_q} L_{q-1} \xrightarrow{\varphi_{q-1}} \dots \xrightarrow{\varphi_2} L_1 \xrightarrow{\varphi_1} L_0 \xrightarrow{\varphi_0} M \rightarrow 0$$

be a finite free resolution of  $M$  computed by running **Algorithm-FRES**. Then the next algorithm computes  $\text{p.dim}_A M$  and, meanwhile, checks whether  $M$  is projective or not.

---

**Algorithm-p.dim**

INPUT  $\mathcal{L}_\bullet$  the given finite free resolution ;  $q$  the length of  $\mathcal{L}_\bullet$

OUTPUT  $\text{p.dim}_A M$

INITIALIZATION  $i := q$

IF  $i = 0$  THEN

$\text{p.dim}_A M := 0$

ELSE

```

LOOP
  use Proposition 3.3.6 to check the invertibility of
  the matrix  $Q_{\varphi_i}$  of  $\varphi_i$ 
  IF  $Q_{\varphi_i}$  is not right invertible THEN
    p.dim $_A M := i$ 
  ELSE
     $i := i - 1$ 
    IF  $i = 0$  THEN
      p.dim $_A M := 0$ 
    ELSE
       $\mathcal{L}_\bullet := (0 \rightarrow L_{i-1} \xrightarrow{\varphi'_{i-1}} L'_{i-2} \xrightarrow{\varphi'_{i-2}} L_{i-3} \xrightarrow{\varphi_{i-3}} \cdots \xrightarrow{\varphi_1} L_0 \xrightarrow{\varphi_0} M \rightarrow 0$ 
      in which  $L'_{i-2} = L_i \oplus L_{i-2}$ , the homomorphism  $\varphi'_{i-1} = \psi$ 
      is computed by using Proposition 3.3.4 (or its proof), and
       $\varphi'_{i-2}$  is defined before the theorem.)
    END
  END
END
END
END

```

---

## 4. Minimal Finite Graded Free Resolutions

In this chapter we demonstrate how the methods and algorithms, developed in ([CDNR], [KR2]) for computing minimal homogeneous generating sets of graded submodules and graded quotient modules of free modules over commutative polynomial algebras, can be adapted for computing minimal homogeneous generating sets of graded submodules and graded quotient modules of free modules over an  $\mathbb{N}$ -graded solvable polynomial  $K$ -algebras  $A$  with the degree-0 homogeneous part  $A_0 = K$ , and how the algorithmic procedures of computing minimal graded free resolutions for finitely generated modules over  $A$  can be achieved.

In the literature, a finitely generated  $\mathbb{N}$ -graded  $K$ -algebra  $A = \bigoplus_{p \in \mathbb{N}} A_p$  with the degree-0 homogeneous part  $A_0 = K$  is usually referred to as a *connected  $\mathbb{N}$ -graded  $K$ -algebra*. Concerning introductions to minimal resolutions of graded modules over a (commutative or noncommutative) connected  $\mathbb{N}$ -graded  $K$ -algebra (or more generally an  $\mathbb{N}$ -graded local  $K$ -algebra) and relevant results, one may refer to ([Eis], Chapter 19), ([Kr1], Chapter 3), and [Li3].

All notions, notations and conventions introduced before are maintained.



### 4.1. $\mathbb{N}$ -graded Solvable Polynomial Algebras

In this section we specify, by means of positive-degree functions (as defined in Section 1.1 of Chapter 1), the structure of  $\mathbb{N}$ -graded solvable polynomial  $K$ -algebras with the degree-0 homogeneous part equal to  $K$ .

For convenience, we first recall that the condition (S2) given in (Definition 1.1.3 of Chapter 1) is equivalent to

- (S2') There is a monomial ordering  $\prec$  on  $\mathcal{B}$ , i.e.,  $(\mathcal{B}, \prec)$  is an admissible system of  $A$ , such that for all generators  $a_i, a_j$  of  $A$  with  $1 \leq i < j \leq n$ ,

$$\begin{aligned} a_j a_i &= \lambda_{ji} a_i a_j + f_{ji} \\ &\text{where } \lambda_{ji} \in K^*, f_{ji} = \sum \mu_k a^{\alpha(k)} \in K\text{-span}\mathcal{B} \\ &\text{with } \mathbf{LM}(f_{ji}) \prec a_i a_j \text{ if } f_{ji} \neq 0. \end{aligned}$$

Now, let  $A = K[a_1, \dots, a_n]$  be a solvable polynomial  $K$ -algebra with admissible system  $(\mathcal{B}, \prec)$ , where  $\mathcal{B} = \{a^\alpha = a_1^{\alpha_1} \cdots a_n^{\alpha_n} \mid \alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n\}$  is the PBW  $K$ -basis of  $A$  and  $\prec$  is a monomial ordering on  $\mathcal{B}$ . Suppose that  $A$  is an  $\mathbb{N}$ -graded algebra with the degree-0 homogeneous part equal to  $K$ , namely  $A = \bigoplus_{p \in \mathbb{N}} A_p$ , where the degree- $p$  homogeneous part  $A_p$  is a  $K$ -subspace of  $A$ ,  $A_0 = K$ , and  $A_{p_1} A_{p_2} \subseteq A_{p_1+p_2}$  for all  $p_1, p_2 \in \mathbb{N}$ . Then, since conventionally any generator  $a_i$  of  $A$  is not contained in the ground field  $K$ , writing  $d_{\text{gr}}(f) = p$  for the *graded-degree* (abbreviated to *gr-degree*) of a nonzero homogeneous element  $f \in A_p$ , we have

$$d_{\text{gr}}(a_i) = m_i, \quad 1 \leq i \leq n,$$

for some positive integers  $m_i$ . It turns out that if  $a^\alpha = a_1^{\alpha_1} \cdots a_n^{\alpha_n} \in \mathcal{B}$ , then  $d_{\text{gr}}(a^\alpha) = \sum_{i=1}^n \alpha_i m_i$ . This shows that  $d_{\text{gr}}(\ )$  gives rise to a positive-degree function on  $A$  as defined in (Section 1.1 of Chapter 1), such that

- (1)  $A_p = K\text{-span}\{a^\alpha \in \mathcal{B} \mid d_{\text{gr}}(a^\alpha) = p\}$ ,  $p \in \mathbb{N}$ ;
- (2) for  $1 \leq i < j \leq n$ , all the relations  $a_j a_i = \lambda_{ji} a_i a_j + f_{ji}$  with  $f_{ji} = \sum \mu_k a^{\alpha(k)}$  presented in (S2') above, satisfy  $d_{\text{gr}}(a^{\alpha(k)}) = d_{\text{gr}}(a_i a_j)$  whenever  $\mu_k \neq 0$ .

Conversely, given a positive-degree function  $d(\ )$  on  $A$  (as defined in Section 1.1 of Chapter 1) such that  $d(a_i) = m_i > 0$ ,  $1 \leq i \leq n$ , then we know that  $A$  has an  $\mathbb{N}$ -graded  $K$ -module structure, i.e.,  $A = \bigoplus_{p \in \mathbb{N}} A_p$  with

$A_p = K\text{-span}\{a^\alpha \in \mathcal{B} \mid d(a^\alpha) = p\}$ , in particular,  $A_0 = K$ . It is straightforward to verify that if furthermore for  $1 \leq i < j \leq n$ , all the relations  $a_j a_i = \lambda_{ji} a_i a_j + f_{ji}$  with  $f_{ji} = \sum \mu_k a^{\alpha(k)}$  presented in (S2') above, satisfy  $d_{\text{gr}}(a^{\alpha(k)}) = d_{\text{gr}}(a_i a_j)$  whenever  $\mu_k \neq 0$ , then  $A_{p_1} A_{p_2} \subseteq A_{p_1+p_2}$  holds for all  $p_1, p_2 \in \mathbb{N}$ , i.e.,  $A$  is turned into an  $\mathbb{N}$ -graded solvable polynomial algebra with the degree-0 homogeneous part  $A_0 = K$ .

Summing up, we have reached the following

**4.1.1. Proposition** A solvable polynomial algebra  $A = K[a_1, \dots, a_n]$  is an  $\mathbb{N}$ -graded algebra with the degree-0 homogeneous part  $A_0 = K$  if and only if there is a positive-degree function  $d(\ )$  on  $A$  (as defined in Section 1.1 of Chapter 1) such that for  $1 \leq i < j \leq n$ , all the relations  $a_j a_i = \lambda_{ji} a_i a_j + f_{ji}$  with  $f_{ji} = \sum \mu_k a^{\alpha(k)}$  presented in (S2') above, satisfy  $d_{\text{gr}}(a^{\alpha(k)}) = d_{\text{gr}}(a_i a_j)$  whenever  $\mu_k \neq 0$ . □

To make the compatibility with the structure of  $\mathbb{N}$ -filtered solvable polynomial algebras specified in Chapter 4, it is necessary to emphasize the role played by a positive-degree function in the structure of  $\mathbb{N}$ -graded solvable polynomial algebras we specified in this section, that is, from now on in the rest of this paper we keep using the following

**Convention** An  $\mathbb{N}$ -graded solvable polynomial  $K$ -algebra  $A = \bigoplus_{p \in \mathbb{N}} A_p$  with  $A_0 = K$  is always referred to as an  *$\mathbb{N}$ -graded solvable polynomial algebra with respect to a positive-degree function  $d(\ )$* .

**Remark** Let  $A = K[a_1, \dots, a_n] = \bigoplus_{p \in \mathbb{N}} A_p$  be an  $\mathbb{N}$ -graded solvable polynomial algebra with respect to a positive-degree function  $d(\ )$ .

(i) We emphasize that *every*  $a^\alpha \in \mathcal{B}$  is a homogeneous elements of  $A$  and  $d(a^\alpha) = d_{\text{gr}}(a^\alpha)$ , where  $d_{\text{gr}}(\ )$ , as we defined above, is the gr-degree function on nonzero homogeneous elements of  $A$ .

(ii) Since  $A$  is a domain (Theorem 1.2.3), the gr-degree function  $d_{\text{gr}}(\ )$  has the property that for all nonzero homogeneous elements  $h_1, h_2 \in A$ ,

$$(\mathbb{P}1) \quad d_{\text{gr}}(h_1 h_2) = d_{\text{gr}}(h_1) + d_{\text{gr}}(h_2).$$

From now on we shall freely use this property without additional indication.

Typical noncommutative  $\mathbb{N}$ -graded solvable polynomial algebras are provided by the multiplicative analogues  $\mathcal{O}_n(\lambda_{ji})$  of the Weyl algebra (see Example (4) given in Section 1.1 of Chapter 1), where the positive-degree function on  $\mathcal{O}_n(\lambda_{ji})$  can be defined by setting  $d(x_i) = m_i$  for any fixed tuple  $(m_1, \dots, m_n)$  of positive integers.

Another family of noncommutative  $\mathbb{N}$ -graded solvable polynomial algebras are provided by the algebras  $M_q(2, K)$  of  $2 \times 2$  quantum matrices (see Example (7) given in Section 1.1 of Chapter 1), where each generator is assigned the degree 1. More generally, let  $\Lambda = (\lambda_{ij})$  be a multiplicatively antisymmetric  $n \times n$  matrix over  $K$ , and let  $\lambda \in K^*$  with  $\lambda \neq -1$ . Considering the multiparameter coordinate ring of quantum  $n \times n$  matrices over  $K$  (see [Good]), namely the  $K$ -algebra  $\mathcal{O}_{\lambda, \Lambda}(M_n(K))$  generated by  $n^2$  elements  $a_{ij}$  ( $1 \leq i, j \leq n$ ) subject to the relations

$$a_{\ell m} a_{ij} = \begin{cases} \lambda_{\ell i} \lambda_{jm} a_{ij} a_{\ell m} + (\lambda - 1) \lambda_{\ell i} a_{im} a_{\ell j} & (\ell > i, m > j) \\ \lambda \lambda_{\ell i} \lambda_{jm} a_{ij} a_{\ell m} & (\ell > i, m \leq j) \\ \lambda_{jm} a_{ij} a_{\ell m} & (\ell = i, m > j) \end{cases}$$

Then  $\mathcal{O}_{\lambda, \Lambda}(M_n(K))$  is an  $\mathbb{N}$ -graded solvable polynomial algebra, where each generator has degree 1.

Moreover, by ([LW], [Lil], or the later Chapter 4), the associated graded algebra and the Rees algebra of every  $\mathbb{N}$ -filtered solvable polynomial algebra with a graded monomial ordering  $\prec_{gr}$  are  $\mathbb{N}$ -graded solvable polynomial algebras of the type we specified in this section.

Finally, we point out that if a solvable polynomial algebra  $A = K[a_1, \dots, a_n]$  has a graded monomial ordering  $\prec_{gr}$  with respect to some given positive-degree function  $d(\ )$  (see Section 1.1 of Chapter 1), then, by Proposition 4.1.1, it is easy to check whether  $A$  is an  $\mathbb{N}$ -graded algebra with respect to  $d(\ )$  or not.

## 4.2. $\mathbb{N}$ -Graded Free Modules

Let  $A = K[a_1, \dots, a_n]$  be an  $\mathbb{N}$ -graded solvable polynomial algebra with respect to a positive-degree function  $d(\ )$ , and let  $(\mathcal{B}, \prec)$  be an admissible system of  $A$ . In this section we demonstrate how to construct  $\mathbb{N}$ -graded free left modules over  $A$  and, furthermore, we highlight that if the input data is a finite set of nonzero homogeneous elements, then **Algorithm-LGB** (presented in Theorem 2.3.4 of Chapter 2) produces a homogeneous

left Gröbner bases for the graded submodule generated by the given homogeneous elements.

We start by a little generality of  $\mathbb{N}$ -graded modules. Let  $M$  be a left  $A$ -module. If  $M = \bigoplus_{q \in \mathbb{N}} M_q$  with each  $M_q$  a  $K$ -subspace of  $M$ , such that  $A_p M_q \subseteq M_{p+q}$  for all  $p, q \in \mathbb{N}$ , then  $M$  is called an  $\mathbb{N}$ -graded  $A$ -module. For each  $q \in \mathbb{N}$ , nonzero elements in  $M_q$  are called *homogeneous elements of degree  $q$* , and accordingly  $M_q$  is called the *degree- $q$  homogeneous part* of  $M$ . If  $\xi \in M_q$  and  $\xi \neq 0$ , then we write  $d_{\text{gr}}(\xi)$  for the *graded-degree* (abbreviated to *gr-degree*) of  $\xi$  as a homogeneous element of  $M$ , i.e.,  $d_{\text{gr}}(\xi) = q$ .

Let  $M = \bigoplus_{q \in \mathbb{N}} M_q$  be a nonzero  $\mathbb{N}$ -graded  $A$ -module, and  $T$  a subset of homogeneous elements of  $M$ . If  $T$  generates  $M$ , i.e.,  $M = \sum_{\xi \in T} A\xi$ , then  $T$  is called a *homogeneous generating set* of  $M$ . Clearly, if  $T = \{\xi_i \mid i \in I\}$  is a homogeneous generating set of  $M$  with  $d_{\text{gr}}(\xi_i) = b_i$  for  $\xi_i \in T$ , then  $M_q = \sum_{p_i + b_i = q} A_{p_i} \xi_i$  for all  $q \in \mathbb{N}$ .

If a submodule  $N$  of the  $\mathbb{N}$ -graded  $A$ -module  $M = \bigoplus_{q \in \mathbb{N}} M_q$  is generated by homogeneous elements, i.e.,  $N$  has a homogeneous generating set, then  $N$  is called a *graded submodule* of  $M$ . A graded submodule  $N$  has the  $\mathbb{N}$ -graded structure  $N = \bigoplus_{q \in \mathbb{N}} N_q$  with  $N_q = N \cap M_q$ , such that  $A_p N_q \subseteq N_{p+q}$  for all  $p, q \in \mathbb{N}$ .

With the graded submodule  $N$  of  $M$  as described above, the quotient module  $M/N$  is an  $\mathbb{N}$ -graded  $A$ -module with the  $\mathbb{N}$ -graded structure  $M/N = \bigoplus_{q \in \mathbb{N}} (M/N)_q$ , where for each  $q \in \mathbb{N}$ ,  $(M/N)_q = (M_q + N)/N$ . Indeed, a submodule  $N$  of  $M$  is a graded submodule if and only if the quotient module  $M/N$  is an  $\mathbb{N}$ -graded  $A$ -module with the  $\mathbb{N}$ -graded structure  $M/N = \bigoplus_{q \in \mathbb{N}} (M_q + N)/N$ .

Now, let  $L = \bigoplus_{i=1}^s A e_i$  be a free left  $A$ -module with the  $A$ -basis  $\{e_1, \dots, e_s\}$ . Then  $L$  has the  $K$ -basis  $\mathcal{B}(e) = \{a^\alpha e_i \mid a^\alpha \in \mathcal{B}, 1 \leq i \leq s\}$  and, for an *arbitrarily* fixed  $\{b_1, \dots, b_s\} \subset \mathbb{N}$ , one checks that  $L$  can be turned into an  $\mathbb{N}$ -graded free  $A$ -module  $L = \bigoplus_{q \in \mathbb{N}} L_q$  by setting

$$L_q = \{0\} \text{ if } q < \min\{b_1, \dots, b_s\}; \text{ otherwise } L_q = \sum_{p_i + b_i = q} A_{p_i} e_i, \quad q \in \mathbb{N},$$

or alternatively, for  $q \geq \min\{b_1, \dots, b_s\}$ ,

$$L_q = K\text{-span}\{a^\alpha e_i \in \mathcal{B}(e) \mid d(a^\alpha) + b_i = q\}, \quad q \in \mathbb{N},$$

such that  $d_{\text{gr}}(e_i) = b_i$ ,  $1 \leq i \leq s$ .

As with the gr-degree of homogeneous elements in  $A$ , noticing that  $d_{\text{gr}}(a^\alpha e_i) = d(a^\alpha) + b_i$  for all  $a^\alpha e_i \in \mathcal{B}(e)$  and that  $A$  is a domain, from now on we shall freely use the following property without additional indication: for all nonzero homogeneous elements  $h \in A$  and all nonzero homogeneous elements  $\xi \in L$ ,

$$(\mathbb{P}2) \quad d_{\text{gr}}(h\xi) = d_{\text{gr}}(h) + d_{\text{gr}}(\xi).$$

**Remark** Although we have remarked that  $d(a^\alpha) = d_{\text{gr}}(a^\alpha)$  for all  $a^\alpha \in \mathcal{B}$ ,  $d(a^\alpha)$  is used in constructing  $L_q$  just for highlighting the role of  $d(\cdot)$ .

**Convention** Unless otherwise stated, from now on throughout the subsequent texts if we say that  $L$  is an  $\mathbb{N}$ -graded free module over an  $\mathbb{N}$ -graded solvable polynomial algebra  $A$  with respect to a positive-degree function  $d(\cdot)$ , then it always means that  $L$  has an  $\mathbb{N}$ -gradation as constructed above.

Let  $L = \bigoplus_{q \in \mathbb{N}} L_q$  be an  $\mathbb{N}$ -graded free  $A$ -module, and  $N$  a graded submodule of  $L$ . A left Gröbner basis  $\mathcal{G}$  of  $N$  is called a *homogeneous left Gröbner basis* if  $\mathcal{G}$  consists of homogeneous elements.

Note that monomials in  $\mathcal{B}$  are homogeneous elements of  $A$ , thereby left  $S$ -polynomials of homogeneous elements are homogeneous elements, and remainders of homogeneous elements on division by homogeneous remain homogeneous elements. Thus, the following assertion is clear now.

**4.2.1. Theorem** With notation as above, if a graded submodule  $N = \sum_{i=1}^m A\xi_i$  of  $L$  is generated by the set of nonzero homogeneous elements  $\{\xi_1, \dots, \xi_m\}$ , then, with the initial input data  $\{\xi_1, \dots, \xi_m\}$ , **Algorithm-LGB** (presented in Theorem 2.3.4 of Chapter 2) produces a finite homogeneous left Gröbner basis  $\mathcal{G}$  for  $N$  with respect to any given monomial ordering  $\prec_e$  on  $\mathcal{B}(e)$ .

### 4.3. Computation of Minimal Homogeneous Generating Sets

In this section,  $A = K[a_1, \dots, a_n]$  denotes an  $\mathbb{N}$ -graded solvable polynomial algebra with respect to a positive-degree function  $d(\cdot)$ ,  $(\mathcal{B}, \prec)$  denotes

a fixed admissible system of  $A$ ,  $L = \bigoplus_{i=1}^s Ae_i$  denotes an  $\mathbb{N}$ -graded free  $A$ -module such that  $d_{\text{gr}}(e_i) = b_i$ ,  $1 \leq i \leq s$ , and  $\prec_e$  denotes a fixed left monomial ordering on the  $K$ -basis  $\mathcal{B}(e)$  of  $L$ . Moreover, as before we write  $S_\ell(\xi_i, \xi_j)$  for the left S-polynomial of two elements  $\xi_i, \xi_j \in L$ .

Let  $N$  be a finitely generated graded submodule  $N$  of  $L$ . With the preparation made in the previous two sections, our aim of the current section is to provide an algorithmic way of computing

- (1) a minimal homogeneous generating set of  $N$ , and
- (2) a minimal homogeneous generating set of the graded quotient module  $M = L/N$ .

The argumentation we are going to present below is similar to the commutative case (cf. [KR], Section 4.5, Section 4.7). To better understand why the similar argumentation can go through the noncommutative case, let us again remind that although monomials from the PBW  $K$ -basis  $\mathcal{B}$  of  $A$  *can no longer behave as well as monomials in a commutative polynomial algebra* (namely the product of two monomials is not necessarily a monomial), every monomial from  $\mathcal{B}$  is a homogeneous element in the  $\mathbb{N}$ -graded structure of  $A$  (as we remarked in section 4.1), thereby the product of two monomials is a homogeneous element. Bearing in mind this fact, one will see that the rule of division, Proposition 1.1.4(i) (Section 1 of Chapter 1), Lemma 2.1.2(ii) (Section 1 of Chapter 2), and the properties (P1), (P2) mentioned in previous Section 1 and Section 2 respectively, all together make the work done.

We start by a detailed discussion on computing  $n$ -truncated left Gröbner bases for graded submodules of  $L$ . Except for helping us to compute a minimal homogeneous generating set, from the definition and the characterization given below one may see clearly that having an  $n$ -truncated left Gröbner basis will be very useful in dealing with certain problems involving only degree- $n$  homogeneous elements.

**4.3.1. Definition** Let  $G = \{g_1, \dots, g_t\}$  be a subset of nonzero homogeneous elements of  $L$ ,  $N = \sum_{i=1}^t Ag_i$  the graded submodule generated by  $G$ , and let  $n \in \mathbb{N}$ ,  $G_{\leq n} = \{g_j \in G \mid d_{\text{gr}}(g_j) \leq n\}$ . If, for each nonzero homogeneous element  $\xi \in N$  with  $d_{\text{gr}}(\xi) \leq n$ , there is some  $g_i \in G_{\leq n}$  such that  $\mathbf{LM}(g_i) \mid \mathbf{LM}(\xi)$  with respect to  $\prec_e$ , then we call  $G_{\leq n}$  an  *$n$ -truncated left Gröbner basis* of  $N$  with respect to  $(\mathcal{B}(e), \prec_e)$ .

By the definition above, the lemma below is straightforward.

**4.3.2. Lemma** Let  $\mathcal{G} = \{g_1, \dots, g_t\}$  be a homogeneous left Gröbner basis for the graded submodule  $N = \sum_{i=1}^t Ag_i$  of  $L$  with respect to  $(\mathcal{B}(e), \prec_e)$ . For each  $n \in \mathbb{N}$ , put  $\mathcal{G}_{\leq n} = \{g_j \in \mathcal{G} \mid d_{\text{gr}}(g_j) \leq n\}$ ,  $N_{\leq n} = \cup_{q=0}^n N_q$  where each  $N_q$  is the degree- $q$  homogeneous part of  $N$ , and let  $N(n) = \sum_{\xi \in N_{\leq n}} A\xi$  be the graded submodule generated by  $N_{\leq n}$ . The following statements hold.

- (i)  $\mathcal{G}_{\leq n}$  is an  $n$ -truncated left Gröbner basis of  $N$ . Thus, if  $\xi \in L$  is a homogeneous element with  $d_{\text{gr}}(\xi) \leq n$ , then  $\xi \in N$  if and only if  $\bar{\xi}^{\mathcal{G}_{\leq n}} = 0$ , i.e.,  $\xi$  is reduced to zero on division by  $\mathcal{G}_{\leq n}$ .
- (ii)  $N(n) = \sum_{g_j \in \mathcal{G}_{\leq n}} Ag_j$ , and  $\mathcal{G}_{\leq n}$  is an  $n$ -truncated left Gröbner basis of  $N(n)$ .

**Proof** Exercise. □

In light of Theorem 2.3.3 and Theorem 2.3.4 (presented in Section 3 of Chapter 2), an  $n$ -truncated left Gröbner basis is characterized as follows.

**4.3.3. Proposition** Let  $N = \sum_{i=0}^s Ag_i$  be the graded submodule of  $L$  generated by a set of homogeneous elements  $G = \{g_1, \dots, g_m\}$ . For each  $n \in \mathbb{N}$ , put  $G_{\leq n} = \{g_j \in G \mid d_{\text{gr}}(g_j) \leq n\}$ . The following statements are equivalent with respect to the given  $(\mathcal{B}(e), \prec_e)$ .

- (i)  $G_{\leq n}$  is an  $n$ -truncated left Gröbner basis of  $N$ .
- (ii) Every nonzero left S-polynomial  $S_{\ell}(g_i, g_j)$  of  $d_{\text{gr}}(S_{\ell}(g_i, g_j)) \leq n$  is reduced to zero on division by  $G_{\leq n}$ , i.e.,  $\overline{S_{\ell}(g_i, g_j)}^{G_{\leq n}} = 0$ .

**Proof** Recall that if  $g_i, g_j \in G$ ,  $\mathbf{LT}(g_i) = \lambda_i a^{\alpha} e_t$  with  $\alpha = (\alpha_1, \dots, \alpha_n)$ ,  $\mathbf{LT}(g_j) = \lambda_j a^{\beta} e_t$  with  $\beta = (\beta_1, \dots, \beta_n)$ , and  $\gamma = (\gamma_1, \dots, \gamma_n)$  with  $\gamma_i = \max\{\alpha_i, \beta_i\}$ ,  $1 \leq i \leq n$ , then

$$S_{\ell}(g_i, g_j) = \frac{1}{\mathbf{LC}(a^{\gamma-\alpha} g_i)} a^{\gamma-\alpha} g_i - \frac{1}{\mathbf{LC}(a^{\gamma-\beta} g_j)} a^{\gamma-\beta} g_j$$

is a homogeneous element in  $N$  with  $d_{\text{gr}}(S_{\ell}(g_i, g_j)) = d(a^{\gamma}) + b_t$  by the foregoing property ( $\mathbb{P}4$ ). If  $d_{\text{gr}}(S_{\ell}(g_i, g_j)) \leq n$ , then it follows from (i) that (ii) holds.

Conversely, suppose that (ii) holds. To see that  $G_{\leq n}$  is an  $n$ -truncated left Gröbner basis of  $N$ , let us run **Algorithm-LGB** (presented in Theorem 2.3.4 of Chapter 2) with the initial input data  $G$ . Without

optimizing **Algorithm-LGB** we may certainly assume that  $G \subseteq \mathcal{G}$ , thereby  $G_{\leq n} \subseteq \mathcal{G}_{\leq n}$  where  $\mathcal{G}$  is the new input set returned by each pass through the WHILE loop. On the other hand, by the construction of  $S_\ell(g_i, g_j)$  and the property (P2) given in the last section, we know that if  $d_{\text{gr}}(S_\ell(g_i, g_j)) \leq n$ , then  $d_{\text{gr}}(g_i) \leq n$ ,  $d_{\text{gr}}(g_j) \leq n$ . Hence, the assumption (ii) implies that **Algorithm-LGB** does not append any new element of degree  $\leq n$  to  $\mathcal{G}$ . Therefore,  $G_{\leq n} = \mathcal{G}_{\leq n}$ . By Lemma 4.3.2 we conclude that  $G_{\leq n}$  is an  $n$ -truncated left Gröbner basis of  $N$ .  $\square$

**4.3.4. Corollary** Let  $N = \sum_{i=1}^m Ag_i$  be the graded submodule of  $L$  generated by a set of homogeneous elements  $G = \{g_1, \dots, g_m\}$ . Suppose that  $G_{\leq n} = \{g_j \in G \mid d_{\text{gr}}(g_j) \leq n\}$  is an  $n$ -truncated left Gröbner basis of  $N$  with respect to  $(\mathcal{B}(e), \prec_e)$ .

(i) If  $\xi \in L$  is a nonzero homogeneous element of  $d_{\text{gr}}(\xi) = n$  such that  $\mathbf{LM}(g_i) \nparallel \mathbf{LM}(\xi)$  for all  $g_i \in G_{\leq n}$ , then  $G' = G_{\leq n} \cup \{\xi\}$  is an  $n$ -truncated left Gröbner basis for both the graded submodules  $N' = N + A\xi$  and  $N'' = \sum_{g_j \in G_{\leq n}} Ag_j + A\xi$  of  $L$ .

(ii) If  $n \leq n_1$  and  $\xi \in L$  is a nonzero homogeneous element of  $d_{\text{gr}}(\xi) = n_1$  such that  $\mathbf{LM}(g_i) \nparallel \mathbf{LM}(\xi)$  for all  $g_i \in G_{\leq n}$ , then  $G' = G_{\leq n} \cup \{\xi\}$  is an  $n_1$ -truncated left Gröbner basis for the graded submodule  $N' = \sum_{g_j \in G_{\leq n}} Ag_j + A\xi$  of  $L$ .

**Proof** If  $\xi \in L$  is a nonzero homogeneous element of  $d_{\text{gr}}(\xi) = n_1 \geq n$  and  $\mathbf{LM}(\xi_i) \nparallel \mathbf{LM}(\xi)$  for all  $\xi_i \in G_{\leq n}$ , then noticing the property mentioned in (Lemma 2.1.2(ii) of Chapter 2) and the property (P2) mentioned in previous Section 2, we see that every nonzero left S-polynomial  $S_\ell(\xi, \xi_i)$  with  $\xi_i \in G$  has  $d_{\text{gr}}(S_\ell(\xi, \xi_i)) > n$ . Hence both (i) and (ii) hold by Proposition 4.3.3.  $\square$

Based on the discussion above, the next proposition tells us that **Algorithm-LGB** (presented in Theorem 2.3.4 of Chapter 2) can be modified for computing  $n$ -truncated left Gröbner bases.

**4.3.5. Proposition** (Compare with ([KR2], Proposition 4.5.10)) Given a finite set of nonzero homogeneous elements  $U = \{\xi_1, \dots, \xi_m\} \subset L$  with  $d_{\text{gr}}(\xi_1) \leq d_{\text{gr}}(\xi_2) \leq \dots \leq d_{\text{gr}}(\xi_m)$ , and a positive integer  $n_0 \geq d_{\text{gr}}(\xi_1)$ , the following algorithm computes an  $n_0$ -truncated left Gröbner basis  $\mathcal{G} = \{g_1, \dots, g_t\}$  for the graded submodule  $N = \sum_{i=1}^m A\xi_i$  such that  $d_{\text{gr}}(g_1) \leq$



$$d_{\text{gr}}(g_2) \leq \cdots d_{\text{gr}}(g_t).$$

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**Algorithm-TRUNC**


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INPUT:  $U = \{\xi_1, \dots, \xi_m\}$  with  $d_{\text{gr}}(\xi_1) \leq d_{\text{gr}}(\xi_2) \leq \cdots d_{\text{gr}}(\xi_m)$   
 $n_0$ , where  $n_0 \geq d_{\text{gr}}(\xi_1)$

OUTPUT:  $\mathcal{G} = \{g_1, \dots, g_t\}$  an  $n_0$ -truncated left Gröbner basis of  $N$

INITIALIZATION:  $\mathcal{S}_{\leq n_0} := \emptyset$ ,  $W := U$ ,  $\mathcal{G} := \emptyset$ ,  $t' := 0$

LOOP

$n := \min\{d_{\text{gr}}(\xi_i), d_{\text{gr}}(S_\ell(g_i, g_j)) \mid \xi_i \in W, S_\ell(g_i, g_j) \in \mathcal{S}_{\leq n_0}\}$

$\mathcal{S}_n := \{S_\ell(g_i, g_j) \in \mathcal{S}_{\leq n_0} \mid d_{\text{gr}}(S_\ell(g_i, g_j)) = n\}$

$W_n := \{\xi_j \in W \mid d_{\text{gr}}(\xi_j) = n\}$

$\mathcal{S}_{\leq n_0} := \mathcal{S}_{\leq n_0} - \mathcal{S}_n$ ,  $W := W - W_n$

WHILE  $\mathcal{S}_n \neq \emptyset$  DO

Choose any  $S_\ell(g_i, g_j) \in \mathcal{S}_n$

$\mathcal{S}_n := \mathcal{S}_n - \{S_\ell(g_i, g_j)\}$

IF  $\overline{S_\ell(g_i, g_j)}^{\mathcal{G}} = \eta \neq 0$  with  $\mathbf{LM}(\eta) = a^\rho e_k$  THEN

$t' := t' + 1$ ,  $g_{t'} := \eta$

$\mathcal{S}_{\leq n_0} := \mathcal{S}_{\leq n_0} \cup \left\{ S_\ell(g_i, g_{t'}) \mid \begin{array}{l} g_i \in \mathcal{G}, 1 \leq i < t', \mathbf{LM}(g_i) = a^\tau e_k, \\ 0 < d_{\text{gr}}(S_\ell(g_i, g_{t'})) \leq n_0 \end{array} \right\}$

$\mathcal{G} := \mathcal{G} \cup \{g_{t'}\}$

END

END

WHILE  $W_n \neq \emptyset$  DO

Choose any  $\xi_j \in W_n$

$W_n := W_n - \{\xi_j\}$

IF  $\overline{\xi_j}^{\mathcal{G}} = \eta \neq 0$  with  $\mathbf{LM}(\eta) = a^\rho e_k$  THEN

$t' := t' + 1$ ,  $g_{t'} := \eta$

$\mathcal{S}_{\leq n_0} := \mathcal{S}_{\leq n_0} \cup \left\{ S_\ell(g_i, g_{t'}) \mid \begin{array}{l} g_i \in \mathcal{G}, 1 \leq i < t', \mathbf{LM}(g_i) = a^\tau e_k, \\ 0 < d_{\text{gr}}(S_\ell(g_i, g_{t'})) \leq n_0 \end{array} \right\}$

$\mathcal{G} := \mathcal{G} \cup \{g_{t'}\}$

END

END

UNTIL  $\mathcal{S}_{\leq n_0} = \emptyset$

END

---

**Proof** First note that both the WHILE loops append new elements to  $\mathcal{G}$  by taking the nonzero normal remainders on division by  $\mathcal{G}$ . Thus, with a fixed  $n$ , by the definition of a left S-polynomial and the normality of

$g_{\nu'} \pmod{\mathcal{G}}$ , it is straightforward to check that in both the WHILE loops every new appended  $S_{\ell}(g_i, g_{\nu'})$  has  $d_{\text{gr}}(S_{\ell}(g_i, g_{\nu'})) > n$ . To proceed, let us write  $N(n)$  for the submodule generated by  $\mathcal{G}$  which is obtained after  $W_n$  is exhausted in the second WHILE loop. If  $n_1$  is the first number after  $n$  such that  $\mathcal{S}_{n_1} \neq \emptyset$ , and for some  $S_{\ell}(g_i, g_j) \in \mathcal{S}_{n_1}$ ,  $\eta = \overline{S_{\ell}(g_i, g_j)}^{\mathcal{G}} \neq 0$  in a certain pass through the first WHILE loop, then we note that this  $\eta$  is still contained in  $N(n)$ . Hence, after  $\mathcal{S}_{n_1}$  is exhausted in the first WHILE loop, the obtained  $\mathcal{G}$  generates  $N(n)$  and  $\mathcal{G}$  is an  $n_1$ -truncated left Gröbner basis of  $N(n)$ . Noticing that the algorithm starts with  $\mathcal{S} = \emptyset$  and  $\mathcal{G} = \emptyset$ , inductively it follows from Proposition 4.3.3 and Corollary 4.3.4 that after  $W_{n_1}$  is exhausted in the second WHILE loop, the obtained  $\mathcal{G}$  is an  $n_1$ -truncated left Gröbner basis of  $N(n_1)$ . Since  $n_0$  is finite and all the generators of  $N$  with  $d_{\text{gr}}(\xi_j) \leq n_0$  are processed through the second WHILE loop, the algorithm terminates and the eventually obtained  $\mathcal{G}$  is an  $n_0$ -truncated left Gröbner basis of  $N$ . Finally, the fact that the degrees of elements in  $\mathcal{G}$  are non-decreasingly ordered follows from the choice of the next  $n$  in the algorithm.  $\square$

Let the data  $(A, \mathcal{B}, \prec)$  and  $(L, \mathcal{B}(e)), \prec_e)$  be as fixed before. Combining the foregoing results, we now proceed to show that the algorithm given in ([KR], Theorem 4.6.3)) can be adapted for computing minimal homogeneous generating sets of graded submodules in free modules over  $A$ .

Let  $N$  be a graded submodule of the  $\mathbb{N}$ -graded free  $A$ -module  $L$  fixed above. We say that a homogeneous generating set  $U$  of  $N$  is a *minimal homogeneous generating set* if any proper subset of  $U$  cannot be a generating set of  $N$ . As preparatory result, we first show that the non-commutative analogue of ([KR], Proposition 4.6.1, Corollary 4.6.2) holds true for  $N$ .

**4.3.6. Proposition** Let  $N = \sum_{i=1}^m A\xi_i$  be the graded submodule of  $L$  generated by a set of homogeneous elements  $U = \{\xi_1, \dots, \xi_m\}$ , where  $d_{\text{gr}}(\xi_1) \leq d_{\text{gr}}(\xi_2) \leq \dots \leq d_{\text{gr}}(\xi_m)$ . Put  $N_1 = \{0\}$ ,  $N_i = \sum_{j=1}^{i-1} A\xi_j$ ,  $2 \leq i \leq m$ . The following statements hold.

- (i)  $U$  is a minimal homogeneous generating set of  $N$  if and only if  $\xi_i \notin N_i$ ,  $1 \leq i \leq m$ .
- (ii) The set  $\overline{U} = \{\xi_k \mid \xi_k \in U, \xi_k \notin N_k\}$  is a minimal homogeneous generating set of  $N$ .

**Proof** (i) If  $U$  is a minimal homogeneous generating set of  $N$ , then clearly  $\xi_i \notin N_i$ ,  $1 \leq i \leq m$ .

Conversely, suppose  $\xi_i \notin N_i$ ,  $1 \leq i \leq m$ . If  $U$  is not a minimal homogeneous generating set of  $N$ , then, there is some  $i$  such that  $N$  is generated by  $\{\xi_1, \dots, \xi_{i-1}, \xi_{i+1}, \dots, \xi_m\}$ , thereby  $\xi_i = \sum_{j \neq i} h_j \xi_j$  for some nonzero homogeneous elements  $h_j \in A$  such that  $d_{\text{gr}}(\xi_i) = d_{\text{gr}}(h_j \xi_j) = d_{\text{gr}}(h_j) + d_{\text{gr}}(\xi_j)$ , where the second equality follows from the foregoing property ( $\mathbb{P}4$ ). Thus  $d_{\text{gr}}(\xi_j) \leq d_{\text{gr}}(\xi_i)$  for all  $j \neq i$ . If  $d_{\text{gr}}(\xi_j) < d_{\text{gr}}(\xi_i)$  for all  $j \neq i$ , then  $\xi_i \in \sum_{j=1}^{i-1} A\xi_j$ , which contradicts the assumption. If  $d_{\text{gr}}(\xi_i) = d_{\text{gr}}(\xi_j)$  for some  $j \neq i$ , then since  $h_j \neq 0$  we have  $h_j \in A_0 - \{0\} = K^*$ . Putting  $i' = \max\{i, j \mid f_j \in K^*\}$ , we then have  $\xi_{i'} \in \sum_{j=1}^{i'-1} A\xi_j$ , which again contradicts the assumption. Hence, under the assumption we conclude that  $U$  is a minimal homogeneous generating set of  $N$ .

(ii) In view of (i), it is sufficient to show that  $\overline{U}$  is a homogeneous generating set of  $N$ . Indeed, if  $\xi_i \in U - \overline{U}$ , then  $\xi_i \in \sum_{j=1}^{i-1} A\xi_j$ . By checking  $\xi_{i-1}$  and so on, it follows that  $\xi_i \in \sum_{\xi_k \in \overline{U}} A\xi_k$ , as desired.  $\square$

**4.3.7. Corollary** Let  $U = \{\xi_1, \dots, \xi_m\}$  be a minimal homogeneous generating set of a graded submodule  $N$  of  $L$ , where  $d_{\text{gr}}(\xi_1) \leq d_{\text{gr}}(\xi_2) \leq \dots \leq d_{\text{gr}}(\xi_m)$ , and let  $\xi \in L - N$  be a homogeneous element with  $d_{\text{gr}}(\xi_m) \leq d_{\text{gr}}(\xi)$ . Then  $\widehat{U} = U \cup \{\xi\}$  is a minimal homogeneous generating set of the graded submodule  $\widehat{N} = N + A\xi$ .  $\square$

We are ready now to reach the following

**4.3.8. Theorem** (Compare with ([KR2], Theorem 4.6.3)) Let  $U = \{\xi_1, \dots, \xi_m\} \subset L$  be a finite set of nonzero homogeneous elements of  $L$  with  $d_{\text{gr}}(\xi_1) \leq d_{\text{gr}}(\xi_2) \leq \dots \leq d_{\text{gr}}(\xi_m)$ . Then the algorithm presented below returns a minimal homogeneous generating set  $U_{\min} = \{\xi_{j_1}, \dots, \xi_{j_r}\} \subset U$  for the graded submodule  $N = \sum_{i=1}^m A\xi_i$ ; and meanwhile it returns a homogeneous left Gröbner basis  $\mathcal{G} = \{g_1, \dots, g_t\}$  for  $N$  such that  $d_{\text{gr}}(g_1) \leq d_{\text{gr}}(g_2) \leq \dots \leq d_{\text{gr}}(g_t)$ .

**Algorithm-MINHGS**


---

INPUT:  $U = \{\xi_1, \dots, \xi_m\}$  with  $d_{\text{gr}}(\xi_1) \leq d_{\text{gr}}(\xi_2) \leq \dots \leq d_{\text{gr}}(\xi_m)$   
 OUTPUT:  $U_{\min} = \{\xi_{j_1}, \dots, \xi_{j_r}\} \subset U$  a minimal homogeneous generating set of  $N$ ;  
 $\mathcal{G} = \{g_1, \dots, g_t\}$  a homogeneous left Gröbner basis of  $N$   
 INITIALIZATION:  $\mathcal{S} := \emptyset$ ,  $W := U$ ,  $\mathcal{G} := \emptyset$ ,  $t' := 0$ ,  $U_{\min} := \emptyset$   
 LOOP  
 $n := \min\{d_{\text{gr}}(\xi_i), d_{\text{gr}}(S_{\ell}(g_i, g_j)) \mid \xi_i \in W, S_{\ell}(g_i, g_j) \in \mathcal{S}\}$   
 $\mathcal{S}_n := \{S_{\ell}(g_i, g_j) \in \mathcal{S} \mid d_{\text{gr}}(S_{\ell}(g_i, g_j)) = n\}$ ,  $W_n := \{\xi_j \in W \mid d_{\text{gr}}(\xi_j) = n\}$   
 $\mathcal{S} := \mathcal{S} - \mathcal{S}_n$ ,  $W := W - W_n$   
 WHILE  $\mathcal{S}_n \neq \emptyset$  DO  
   Choose any  $S_{\ell}(g_i, g_j) \in \mathcal{S}_n$   
 $\mathcal{S}_n := \mathcal{S}_n - \{S_{\ell}(g_i, g_j)\}$   
   IF  $\overline{S_{\ell}(g_i, g_j)}^{\mathcal{G}} = \eta \neq 0$  with  $\mathbf{LM}(\eta) = a^{\rho} e_k$  THEN  
      $t' := t' + 1$ ,  $g_{t'} := \eta$   
      $\mathcal{S} := \mathcal{S} \cup \{S_{\ell}(g_i, g_{t'}) \neq 0 \mid g_i \in \mathcal{G}, 1 \leq i < t', \mathbf{LM}(g_i) = a^{\tau} e_k\}$   
      $\mathcal{G} := \mathcal{G} \cup \{g_{t'}\}$   
   END  
 END  
 WHILE  $W_n \neq \emptyset$  DO  
   Choose any  $\xi_j \in W_n$   
 $W_n := W_n - \{\xi_j\}$   
   IF  $\overline{\xi_j}^{\mathcal{G}} = \eta \neq 0$  with  $\mathbf{LM}(\eta) = a^{\rho} e_k$  THEN  
      $U_{\min} := U_{\min} \cup \{\xi_j\}$   
      $t' := t' + 1$ ,  $g_{t'} := \eta$   
      $\mathcal{S} := \mathcal{S} \cup \{S_{\ell}(g_i, g_{t'}) \neq 0 \mid g_i \in \mathcal{G}, 1 \leq i < t', \mathbf{LM}(g_i) = a^{\tau} e_k\}$   
      $\mathcal{G} := \mathcal{G} \cup \{g_{t'}\}$   
   END  
 END  
 END  
 UNTIL  $\mathcal{S} = \emptyset$   
 END

---

**Proof** Since this algorithm is clearly a variant of **Algorithm-LGB** and **Algorithm-TRUNC** with a minimization procedure which works with the finite set  $U$ , it terminates after a certain integer  $n$  is executed, and the eventually obtained  $\mathcal{G}$  is a homogeneous left Gröbner basis for  $N$  in which the degrees of elements are ordered non-decreasingly. It remains

to prove that the eventually obtained  $U_{\min}$  is a minimal homogeneous generating set of  $N$ .

As in the proof of Proposition 4.3.5, let us first bear in mind that for each  $n$ , in both the WHILE loops every new appended  $S_\ell(g_i, g_{i'})$  has  $d_{\text{gr}}(S_\ell(g_i, g_{i'})) > n$ . Moreover, for convenience, let us write  $\mathcal{G}(n)$  for the  $\mathcal{G}$  obtained after  $\mathcal{S}_n$  is exhausted in the first WHILE loop, and write  $U_{\min}[n]$ ,  $\mathcal{G}[n]$  respectively for the  $U_{\min}$ ,  $\mathcal{G}$  obtained after  $W_n$  is exhausted in the second WHILE loop. Since the algorithm starts with  $\mathcal{O} = \emptyset$  and  $\mathcal{G} = \emptyset$ , if, for a fixed  $n$ , we check carefully how the elements of  $U_{\min}$  are chosen during executing the second WHILE loop, and how the new elements are appended to  $\mathcal{G}$  after each pass through the first or the second WHILE loop, then it follows from Proposition 4.3.3 and Corollary 4.3.4 that after  $W_n$  is exhausted, the obtained  $U_{\min}[n]$  and  $\mathcal{G}[n]$  generate the same module, denoted  $N(n)$ , such that  $\mathcal{G}[n]$  is an  $n$ -truncated left Gröbner basis of  $N(n)$ . We now use induction to show that the eventually obtained  $U_{\min}$  is a minimal homogeneous generating set for  $N$ . If  $U_{\min} = \emptyset$ , then it is a minimal generating set of the zero module. To proceed, we assume that  $U_{\min}[n]$  is a minimal homogeneous generating set for  $N(n)$  after  $W_n$  is exhausted in the second WHILE loop. Suppose that  $n_1$  is the first number after  $n$  such that  $\mathcal{S}_{n_1} \neq \emptyset$ . We complete the induction proof below by showing that  $U_{\min}[n_1]$  is a minimal homogeneous generating set of  $N(n_1)$ .

If in a certain pass through the first WHILE loop,  $\overline{S_\ell(g_i, g_j)}^{\mathcal{G}} = \eta \neq 0$  for some  $S_\ell(g_i, g_j) \in \mathcal{S}_{n_1}$ , then we note that  $\eta \in N(n)$ . It follows that after  $\mathcal{S}_{n_1}$  is exhausted in the first WHILE loop,  $\mathcal{G}(n_1)$  generates  $N(n)$  and  $\mathcal{G}(n_1)$  is an  $n_1$ -truncated left Gröbner basis of  $N(n)$ . Next, assume that  $W_{n_1} = \{\xi_{j_1}, \dots, \xi_{j_s}\} \neq \emptyset$  and that the elements of  $W_{n_1}$  are processed in the given order during executing the second WHILE loop. Since  $\mathcal{G}(n_1)$  is an  $n_1$ -truncated left Gröbner basis of  $N(n)$ , if  $\xi_{j_1} \in W_{n_1}$  is such that  $\overline{\xi_{j_1}}^{\mathcal{G}(n_1)} = \eta_1 \neq 0$ , then  $\xi_{j_1}, \eta_1 \in L - N(n)$ . By Corollary 4.3.4, we conclude that  $\mathcal{G}(n_1) \cup \{\eta_1\}$  is an  $n_1$ -truncated Gröbner basis for the module  $N(n) + A\eta_1$ ; and by Corollary 4.3.7, we conclude that  $U_{\min}[n] \cup \{\xi_{j_1}\}$  is a minimal homogeneous generating set of  $N(n) + A\eta_1$ . Repeating this procedure, if  $\xi_{j_2} \in W_{n_1}$  is such that  $\overline{\xi_{j_2}}^{\mathcal{G}(n_1) \cup \{\eta_1\}} = \eta_2 \neq 0$ , then  $\xi_{j_2}, \eta_2 \in L - (N(n) + A\eta_1)$ . By Corollary 4.3.4, we conclude that  $\mathcal{G}(n_1) \cup \{\eta_1, \eta_2\}$  is an  $n_1$ -truncated left Gröbner basis for the module  $N(n) + A\eta_1 + A\eta_2$ ; and by Corollary 4.3.7, we conclude that  $U_{\min}[n] \cup \{\xi_{j_1}, \xi_{j_2}\}$  is

a minimal homogeneous generating set of  $N(n) + A\eta_1 + A\eta_2$ . Continuing this procedure until  $W_{n_1}$  is exhausted we assert that the returned  $\mathcal{G}[n_1] = \mathcal{G}$  and  $U_{\min}[n_1] = U_{\min}$  generate the same module  $N(n_1)$  and  $\mathcal{G}[n_1]$  is an  $n_1$ -truncated left Gröbner basis of  $N(n_1)$  and  $U_{\min}[n_1]$  is a minimal homogeneous generating set of  $N(n_1)$ , as desired. As all elements of  $U$  are eventually processed by the second WHILE loop, we conclude that the finally obtained  $\mathcal{G}$  and  $U_{\min}$  have the properties:  $\mathcal{G}$  generates the module  $N$ ,  $\mathcal{G}$  is an  $n_0$ -truncated left Gröbner basis of  $N$ , and  $U_{\min}$  is a minimal homogeneous generating set of  $N$ .

□

**Remark.** If we are only interested in getting a minimal homogeneous generating set for the submodule  $N$ , then **Algorithm-MINHGS** can indeed be speed up. More precisely, with

$$d_{\text{gr}}(\xi_1) \leq d_{\text{gr}}(\xi_2) \leq \cdots \leq d_{\text{gr}}(\xi_m) = n_0,$$

it follows from the proof above that if we stop executing the algorithm after  $S_{n_0}$  and  $W_{n_0}$  are exhausted, then the resulted  $U_{\min}[n_0]$  is already the desired minimal homogeneous generating set for  $N$ , while  $\mathcal{G}[n_0]$  is an  $n_0$ -truncated left Gröbner basis of  $N$ .

**4.3.9. Corollary** Let  $U = \{\xi_1, \dots, \xi_m\} \subset L$  be a finite set of nonzero homogeneous elements of  $L$  with  $d_{\text{gr}}(\xi_1) = d_{\text{gr}}(\xi_2) = \cdots = d_{\text{gr}}(\xi_m) = n_0$

(i) If  $U$  satisfies  $\mathbf{LM}(\xi_i) \neq \mathbf{LM}(\xi_j)$  for all  $i \neq j$ , then  $U$  is a minimal homogeneous generating set of the graded submodule  $N = \sum_{i=1}^m A\xi_i$  of  $L$ , and meanwhile  $U$  is an  $n_0$ -truncated left Gröbner basis for  $N$ .

(ii) If  $U$  is a minimal left Gröbner basis of the graded submodule  $N = \sum_{i=1}^m A\xi_i$  (i.e.,  $U$  is a left Grobner basis of  $N$  satisfying  $\mathbf{LM}(\xi_i) \neq \mathbf{LM}(\xi_j)$  for all  $i \neq j$ ), then  $U$  is a minimal homogeneous generating set of  $N$ .

**Proof** By the assumption, it follows from the second WHILE loop of **Algorithm-MINHGS** that  $U_{\min} = U$ .

□

Let  $N$  be an arbitrary nonzero graded submodule of the  $\mathbb{N}$ -graded free  $A$ -module  $L = \oplus_{i=1}^s Ae_i$  with  $d_{\text{gr}}(e_i) = b_i$ ,  $1 \leq i \leq s$ , and consider the graded quotient module  $M = L/N$  (see previous Section 4.2). Our next goal is to compute a minimal homogeneous generating set for  $M$ .

Since  $A$  is Noetherian,  $N$  is a finitely generated graded submodule of  $L_0$ . Let  $N = \sum_{j=1}^m A\xi_j$  be generated by the set of nonzero homogeneous elements  $U = \{\xi_1, \dots, \xi_m\}$ , where  $\xi_\ell = \sum_{k=1}^s f_{k\ell}e_k$  with  $f_{k\ell} \in A$ ,  $1 \leq \ell \leq m$ . Then, every nonzero  $f_{k\ell}$  is a homogeneous element of  $A$  such that  $d_{\text{gr}}(\xi_\ell) = d_{\text{gr}}(f_{k\ell}e_k) = d_{\text{gr}}(f_{k\ell}) + b_k$ , where  $b_k = d_{\text{gr}}(e_k)$ ,  $1 \leq k \leq s$ ,  $1 \leq \ell \leq m$ .

**4.3.10. Lemma** With every  $\xi_\ell = \sum_{i=1}^s f_{i\ell}e_i$  as fixed above,  $1 \leq \ell \leq m$ , if the  $i$ -th coefficient  $f_{ij}$  of some  $\xi_j$  is a nonzero constant, say  $f_{ij} = 1$  without loss of generality, then for each  $\ell = 1, \dots, j-1, j+1, \dots, m$ , the element  $\xi'_\ell = \xi_\ell - f_{i\ell}\xi_j$  does not involve  $e_i$ . Putting  $U' = \{\xi'_1, \dots, \xi'_{j-1}, \xi'_{j+1}, \dots, \xi'_m\}$ , there is a graded  $A$ -module isomorphism  $M' = L'/N' \cong L/N = M$ , where  $L' = \oplus_{k \neq i} Ae_k$  and  $N' = \sum_{\xi'_\ell \in U'} A\xi'_\ell$ .

**Proof** Since  $f_{ij} = 1$  by the assumption, we see that every  $\xi'_\ell = \sum_{k \neq i} (f_{k\ell} - f_{i\ell}f_{kj})e_k$  does not involve  $e_i$ . Let  $U' = \{\xi'_1, \dots, \xi'_{j-1}, \xi'_{j+1}, \dots, \xi'_m\}$  and  $N' = \sum_{\xi'_\ell \in U'} A\xi'_\ell$ . Then  $N' \subset L' = \oplus_{k \neq i} Ae_k$ . Again since  $f_{ij} = 1$ , we have  $d_{\text{gr}}(\xi_j) = d_{\text{gr}}(e_i) = b_i$ . It follows from the property (P4) formulated in Subsection 2.1 that

$$\begin{aligned} d_{\text{gr}}(f_{i\ell}f_{kj}e_k) &= d_{\text{gr}}(f_{i\ell}) + d_{\text{gr}}(f_{kj}e_k) \\ &= d_{\text{gr}}(f_{i\ell}) + d_{\text{gr}}(\xi_j) \\ &= d_{\text{gr}}(f_{i\ell}) + b_i \\ &= d_{\text{gr}}(f_{i\ell}e_i) \\ &= d_{\text{gr}}(\xi_\ell) \\ &= d_{\text{gr}}(f_{k\ell}e_k). \end{aligned}$$

Noticing that  $d_{\text{gr}}(f_{i\ell}\xi_j) = d_{\text{gr}}(f_{i\ell}) + d_{\text{gr}}(\xi_j)$ , this shows that in the representation of  $\xi'_\ell$  every nonzero term  $(f_{k\ell} - f_{i\ell}f_{kj})e_k$  is a homogeneous element of degree  $d_{\text{gr}}(\xi_\ell) = d_{\text{gr}}(f_{i\ell}\xi_j)$ , thereby  $M' = L'/N'$  is a graded  $A$ -module. Note that  $N = N' + A\xi_j$  and that  $\xi_j = e_i + \sum_{k \neq i} f_{kj}e_k$ . Without making confusion, if we use the same notation  $\bar{e}_k$  to denote the coset represented by  $e_k$  in  $M'$  and  $M$  respectively, it is now clear that the desired graded  $A$ -module isomorphism  $M' \xrightarrow{\varphi} M$  is naturally defined by  $\varphi(\bar{e}_k) = \bar{e}_k$ ,  $k = 1, \dots, i-1, i+1, \dots, s$ .  $\square$

Let  $M = L/N$  be as fixed above with  $N$  generated by the set of nonzero homogeneous elements  $U = \{\xi_1, \dots, \xi_m\}$ . Then since  $A$  is  $\mathbb{N}$ -graded with  $A_0 = K$ , it is well known that the homogeneous generating

set  $\overline{E} = \{\overline{e}_1, \dots, \overline{e}_s\}$  of  $M$  is a minimal homogeneous generating set if and only if  $\xi_\ell = \sum_{k=1}^s f_{k\ell} e_k$  implies  $d_{\text{gr}}(f_{k\ell}) > 0$  whenever  $f_{k\ell} \neq 0$ ,  $1 \leq \ell \leq m$ .

**4.3.11. Proposition** (Compare with ([KR2], Proposition 4.7.24)) With notation as fixed above, the algorithm presented below returns a subset  $\{e_{i_1}, \dots, e_{i_{s'}}\} \subset \{e_1, \dots, e_s\}$  and a subset  $V = \{v_1, \dots, v_t\} \subset N \cap L'$  such that  $M \cong L'/N'$  as graded  $A$ -modules, where  $L' = \bigoplus_{q=1}^{s'} A e_{i_q}$  with  $s' \leq s$  and  $N' = \sum_{k=1}^t A v_k$ , and such that  $\{\overline{e}_{i_1}, \dots, \overline{e}_{i_{s'}}\}$  is a minimal homogeneous generating set of  $M$ .

---

**Algorithm-MINHGSQ**


---

INPUT:  $E = \{e_1, \dots, e_s\}$ ;  $U = \{\xi_1, \dots, \xi_m\}$

where  $\xi_\ell = \sum_{k=1}^s f_{k\ell} e_k$  with homogeneous  $f_{k\ell} \in A$ ,  $1 \leq \ell \leq m$

OUTPUT:  $E' = \{e_{i_1}, \dots, e_{i_{s'}}\}$ ;  $V = \{v_1, \dots, v_t\} \subset N \cap L'$ , such that

$v_j = \sum_{q=1}^{s'} h_{qj} e_{i_q} \in L' = \bigoplus_{q=1}^{s'} A e_{i_q}$  with  $h_{qj} \notin K^*$

whenever  $h_{qj} \neq 0$ ,  $1 \leq j \leq t$

INITIALIZATION:  $t := m$ ;  $V := U$ ;  $s' := s$ ;  $E' := E$

BEGIN

WHILE there is a  $v_j = \sum_{k=1}^{s'} f_{kj} e_k \in V$  satisfying

$f_{kj} \notin K^*$  for  $k < i$  and  $f_{ij} \in K^*$  DO

for  $T = \{1, \dots, j-1, j+1, \dots, t\}$  compute

$v'_\ell = v_\ell - \frac{1}{f_{ij}} f_{i\ell} v_j$ ,  $\ell \in T$ ,  $r = \#\{\ell \mid \ell \in T, v'_\ell = 0\}$

$t := t - r - 1$

$V := \{v_\ell = v'_\ell \mid \ell \in T, v'_\ell \neq 0\}$

$= \{v_1, \dots, v_t\}$  (after reordered)

$s' := s' - 1$

$E' := E' - \{e_i\} = \{e_1, \dots, e_{s'}\}$  (after reordered)

END

END

---

**Proof** It is clear that the algorithm is finite. The correctness of the algorithm follows immediately from Lemma 4.3.10 and the remark we made before the proposition.



#### 4.4. Computation of Minimal Finite Graded Free Resolutions

Let  $A = K[a_1, \dots, a_n] = \bigoplus_{p \in \mathbb{N}} A_p$  be an  $\mathbb{N}$ -graded solvable polynomial algebra with respect to a positive-degree function  $d(\cdot)$ , and  $(\mathcal{B}, \prec)$  a fixed admissible system of  $A$ . Let  $M = \bigoplus_{q \in \mathbb{N}} M_q$ ,  $M' = \bigoplus_{q \in \mathbb{N}} M'_q$  be  $\mathbb{N}$ -graded left  $A$ -modules, and  $M \xrightarrow{\varphi} M'$  an  $A$ -module homomorphism. If  $\varphi(M_q) \subseteq M'_q$  for all  $q \in \mathbb{N}$ , then  $\varphi$  is called a *graded homomorphism*. In the literature, such graded homomorphisms are also referred to as graded homomorphisms of degree-0 (cf. [NVO]). By the definition it is clear that the identity map of  $\mathbb{N}$ -graded  $A$ -modules is graded homomorphism, and compositions of graded homomorphisms are graded homomorphisms. Thus, all  $\mathbb{N}$ -graded left  $A$ -modules form a subcategory of the category of left  $A$ -modules, in which morphisms are the graded homomorphisms as defined above. Furthermore, if  $M \xrightarrow{\varphi} M'$  is a graded homomorphism, then one checks that the kernel  $\text{Ker}\varphi$  of  $\varphi$  is a graded submodule of  $M$ , and the image  $\text{Im}\varphi$  of  $\varphi$  is a graded submodule of  $M'$  (See previous Section 4.2). Consequently, the exactness of a sequence  $N \xrightarrow{\varphi} M \xrightarrow{\psi} M'$  of graded homomorphisms in the category of  $\mathbb{N}$ -graded  $A$ -modules is defined as the same as for a sequence of usual  $A$ -module homomorphisms, i.e., the sequence satisfies  $\text{Im}\varphi = \text{Ker}\psi$ . Long exact sequence in the category of  $\mathbb{N}$ -graded  $A$ -modules may be defined in an obvious way.

Since  $A$  is Noetherian and  $A_0 = K$ , it is theoretically well known that up to a graded isomorphism of chain complexes in the category of graded  $A$ -modules, every finitely generated graded  $A$ -module  $M$  has a unique minimal graded free resolution (cf. [Eis], Chapter 19; [Kr1], Chapter 3; [Li3]). Based on previously obtained results, in this section we establish the algorithmic procedures for constructing minimal finite graded free resolutions over  $A$ . All notions, notations and conventions used in previous sections are maintained.

Given a finitely generated  $\mathbb{N}$ -graded  $A$ -module  $M = \sum_{i=1}^s Av_i$  with the set of nonzero homogeneous generators  $\{v_1, \dots, v_s\}$  such that  $d_{\text{gr}}(v_i) = b_i$  for  $1 \leq i \leq s$ , consider the  $\mathbb{N}$ -graded free  $A$ -module  $L_0 = \bigoplus_{i=1}^s Ae_i$  with  $d_{\text{gr}}(e_i) = b_i$ ,  $1 \leq i \leq s$ , as constructed in Section 4.2. Then, under the  $\mathbb{N}$ -graded epimorphism  $L_0 \xrightarrow{\varphi_0} M \rightarrow 0$  defined by  $\varphi_0(e_i) = v_i$ ,  $1 \leq i \leq s$ , there is a graded isomorphism  $M \cong L_0/N$ , where  $N = \text{Ker}\varphi_0$ . Thus, we may identify  $M$  with the graded quotient module

$L_0/N$  and write  $M = L_0/N$ .

Recall from the literature that a *minimal graded free resolution* of  $M$  is an exact sequence by free  $A$ -modules and  $A$ -module homomorphisms

$$\mathcal{L}_\bullet \quad \cdots \xrightarrow{\varphi_{i+1}} L_i \xrightarrow{\varphi_i} \cdots \xrightarrow{\varphi_2} L_1 \xrightarrow{\varphi_1} L_0 \xrightarrow{\varphi_0} M \longrightarrow 0$$

in which each  $L_i$  is an  $\mathbb{N}$ -graded free  $A$ -module with a finite homogeneous  $A$ -basis  $E_i = \{e_{i_1}, \dots, e_{i_{s_i}}\}$ , and each  $\varphi_i$  is a graded homomorphism, such that

- (1)  $\varphi_0(E_0)$  is a minimal homogeneous generating set of  $M$ ,  $\text{Ker}\varphi_0 = N$ , and
- (2) for  $i \geq 1$ ,  $\varphi_i(E_i)$  is a minimal homogeneous generating set of  $\text{Ker}\varphi_{i-1}$ .

**4.4.1. Theorem** With notation as fixed above, suppose that  $N = \sum_{i=1}^m A\xi_i$  with the set of nonzero homogeneous generators  $U = \{\xi_1, \dots, \xi_m\}$ . Then the graded  $A$ -module  $M = L_0/N$  has a minimal graded free resolution of length  $d \leq n$ :

$$\mathcal{L}_\bullet \quad 0 \longrightarrow L_d \xrightarrow{\varphi_d} \cdots \xrightarrow{\varphi_2} L_1 \xrightarrow{\varphi_1} L_0 \xrightarrow{\varphi_0} M \longrightarrow 0$$

which can be constructed by implementing the following procedures:

**Procedure 1.** Run **Algorithm-MINHGSQ** of Proposition 4.3.11 with the initial input data  $E = \{e_1, \dots, e_s\}$  and  $U = \{\xi_1, \dots, \xi_m\}$  to compute a subset  $E' = \{e_{i_1}, \dots, e_{i_{s'}}\} \subset \{e_1, \dots, e_s\}$  and a subset  $V = \{v_1, \dots, v_t\} \subset N \cap L'_0$  such that  $M \cong L'_0/N'$  as graded  $A$ -modules, where  $L'_0 = \bigoplus_{q=1}^{s'} Ae_{i_q}$  with  $s' \leq s$  and  $N' = \sum_{k=1}^t Av_k$ , and such that  $\{\bar{e}_{i_1}, \dots, \bar{e}_{i_{s'}}\}$  is a minimal homogeneous generating set of  $M$ .

For convenience, after accomplishing Procedure 1 we may assume that  $E = E'$ ,  $U = V$  and  $N = N'$ . Accordingly we have the short exact sequence

$$0 \longrightarrow N \longrightarrow L_0 \xrightarrow{\varphi_0} M \longrightarrow 0$$

such that  $\varphi_0(E) = \{\bar{e}_1, \dots, \bar{e}_s\}$  is a minimal homogeneous generating set of  $M$ .

**Procedure 2.** Choose a left monomial ordering  $\prec_e$  on the  $K$ -basis  $\mathcal{B}(e)$  of  $L_0$  and run **Algorithm-MINHGS** of Theorem 4.3.8 with the initial input data  $U = \{\xi_1, \dots, \xi_m\}$  to compute a minimal homogeneous generating set  $U_{\min} = \{\xi_{j_1}, \dots, \xi_{j_{s_1}}\}$  and a left Gröbner basis  $\mathcal{G}$  for  $N$ ; at the same time, by keeping track of the reductions during executing the

first WHILE loop and the second WHILE loop respectively, return the matrices  $\mathcal{S}_{r \times t}$  and  $V_{t \times m}$  required by (Theorem 3.1.2, Chapter 3).

**Procedure 3.** By using the division by the left Gröbner basis  $\mathcal{G}$  obtained in Procedure 2, compute the matrix  $U_{m \times t}$  required by (Theorem 3.1.2, Chapter 3). Use the matrices  $\mathcal{S}_{r \times t}$ ,  $V_{t \times m}$  obtained in Procedure 2, the matrix  $U_{m \times t}$  and (Theorem 3.1.2, Chapter 3) to compute a homogeneous generating set of  $N_1 = \text{Syz}(U_{\min})$  in the  $\mathbb{N}$ -graded free  $A$ -module  $L_1 = \bigoplus_{i=1}^{s_1} A\varepsilon_i$ , where the gradation of  $L_1$  is defined by setting  $d_{\text{gr}}(\varepsilon_k) = d_{\text{gr}}(\xi_{j_k})$ ,  $1 \leq k \leq s_1$ .

**Procedure 4.** Construct the exact sequence

$$0 \longrightarrow N_1 \longrightarrow L_1 \xrightarrow{\varphi_1} L_0 \xrightarrow{\varphi_0} M \longrightarrow 0$$

where  $\varphi_1(\varepsilon_k) = \xi_{j_k}$ ,  $1 \leq k \leq s_1$ .

If  $N_1 \neq 0$ , then repeat Procedure 2 – Procedure 4 for  $N_1$  and so on.

Noticing that  $A$  is  $\mathbb{N}$ -graded with the degree-0 homogeneous part  $A_0 = K$ ,  $A$  is Noetherian, and that every finitely generated  $A$ -module  $M$  has finite projective dimension  $\text{p.dim}_A M \leq n$  by (Theorem 3.3.1, Chapter 3), thereby  $A$  is an  $\mathbb{N}$ -graded local ring of finite global homological dimension. It follows from the literature ([Eis], Chapter 19; [Kr1], Chapter 3; [Li3]) that

$$\text{p.dim}_A M = \text{the length of a minimal graded free resolution of } M.$$

Hence, the desired minimal finite graded free resolution  $\mathcal{L}_\bullet$  for  $M$  is then obtained after finite times of processing the above procedures.

## 5. Minimal Finite Filtered Free Resolutions

Recall that the  $\mathbb{N}$ -filtered solvable polynomial algebras with  $\mathbb{N}$ -filtration determined by the natural length of elements from the PBW  $K$ -basis (especially the quadric solvable polynomial algebras are such  $\mathbb{N}$ -filtered algebras) were studied in ([LW], [Li1]). In this chapter, after specifying the  $\mathbb{N}$ -filtered structure determined by a positive-degree function  $d(\cdot)$  on a solvable polynomial algebra  $A = K[a_1, \dots, a_n]$ , we specify the corresponding  $\mathbb{N}$ -filtered structure for free  $A$ -modules. Then, we introduce minimal filtered free resolutions for finitely generated modules over such  $\mathbb{N}$ -filtered algebra  $A$  by introducing minimal F-bases and minimal standard bases for  $A$ -modules and their submodules with respect to good filtration; we show that any two minimal F-bases, respectively any two minimal standard bases, have the same number of elements and the same number of elements of the same filtered-degree, minimal filtered free resolutions are unique up to strict filtered isomorphism of chain complexes in the category of filtered  $A$ -modules, and that minimal filtered free resolutions can be algorithmically computed in case  $A$  has a graded monomial ordering  $\prec_{gr}$ .

Since the standard bases we are going to introduce in terms of good filtration are generalization of classical Macaulay bases (see a remark given in Subsection 3.3), while a classical Macaulay basis  $V$  is characterized in terms of both the leading homogeneous elements (degree forms) of  $V$  and

the homogenized elements of  $V$  (cf. [KR2], P.38, P.55), accordingly, on the basis of Chapter 4, our main idea in reaching the goal of this chapter is to use the filtered-graded transfer strategy (as proposed in [Li1]) by employing both the associated graded algebra (module) and the Rees algebra (module) of an  $\mathbb{N}$ -filtered solvable polynomial algebra (of a filtered module).

All notions, notations and conventions introduced in previous chapters are maintained.

## 5.1. $\mathbb{N}$ -Filtered Solvable Polynomial Algebras

Comparing with the general theory of  $\mathbb{Z}$ -filtered rings [LVO], in this section we formulate the  $\mathbb{N}$ -filtered structure of solvable polynomial algebras determined by means of positive-degree functions.

Let  $A = K[a_1, \dots, a_n]$  be a solvable polynomial algebra with admissible system  $(\mathcal{B}, \prec)$ , where  $\mathcal{B} = \{a^\alpha = a_1^{\alpha_1} \cdots a_n^{\alpha_n} \mid \alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n\}$  is the PBW  $K$ -basis of  $A$  and  $\prec$  is a monomial ordering on  $\mathcal{B}$ , and let  $d(\cdot)$  be a positive-degree function on  $A$  such that  $d(a_i) = m_i > 0$ ,  $1 \leq i \leq n$  (see Section 1.1 of Chapter 1). Put

$$F_p A = K\text{-span}\{a^\alpha \in \mathcal{B} \mid d(a^\alpha) \leq p\}, \quad p \in \mathbb{N},$$

then it is clear that  $F_p A \subseteq F_{p+1} A$  for all  $p \in \mathbb{N}$ ,  $A = \cup_{p \in \mathbb{N}} F_p A$ , and  $1 \in F_0 A = K$ .

**5.1.1. Definition** With notation as above, if  $F_p A F_q A \subseteq F_{p+q} A$  holds for all  $p, q \in \mathbb{N}$ , then we call  $A$  an  *$\mathbb{N}$ -filtered solvable polynomial algebra with respect to the positive-degree function  $d(\cdot)$* , and accordingly we call  $FA = \{F_p A\}_{p \in \mathbb{N}}$  the  *$\mathbb{N}$ -filtration of  $A$  determined by  $d(\cdot)$* .

Note that the  $\mathbb{N}$ -filtration  $FA$  constructed above is clearly *separated* in the sense that if  $f$  is a *nonzero* element of  $L$ , then either  $f \in F_0 A = K$  or  $f \in F_p A - F_{p-1} A$  for some  $p > 0$ . Thus, we may define the *filtered-degree* (abbreviated to *fil-degree*) of a nonzero  $f \in A$ , denoted  $d_{\text{fil}}(f)$ , as follows

$$d_{\text{fil}}(f) = \begin{cases} 0, & \text{if } f \in F_0 A = K, \\ p, & \text{if } f \in F_p A - F_{p-1} A \text{ for some } p > 0. \end{cases}$$

Bearing in mind the definition of  $d_{\text{fil}}(f)$ , the following featured property of  $FA$  will very much help us to deal with the associated graded structures of  $A$  and filtered  $A$ -modules.

**5.1.2. Lemma** If  $f = \sum_i \lambda_i a^{\alpha(i)} \in A$  with  $\lambda_i \in K^*$  and  $a^{\alpha(i)} \in \mathcal{B}$ , then  $d_{\text{fil}}(f) = p$  if and only if  $d(a^{\alpha(i')}) = p$  for some  $i'$  if and only if  $d(f) = p = d_{\text{fil}}(f)$ , where  $d(\ )$  is the given positive-degree function on  $A$ .

**Proof** Exercise.  $\square$

Given a solvable polynomial algebra  $A = K[a_1, \dots, a_n]$  and a positive-degree function  $d(\ )$  on  $A$ , it follows from Definition 1.1.3 (Section 1, Chapter 1), Definition 5.1.1 and Lemma 5.1.2 above that the next proposition is clear now.

**5.1.3. Proposition**  $A$  is an  $\mathbb{N}$ -filtered solvable polynomial algebra with respect to  $d(\ )$  if, for  $1 \leq i < j \leq n$ , all the relations  $a_j a_i = \lambda_{ji} a_i a_j + f_{ji}$  with  $f_{ji} = \sum \mu_k a^{\alpha(k)}$  presented in (S2') (Section 4.1, Chapter 4), satisfy  $d(a^{\alpha(k)}) \leq d(a_i a_j)$  whenever  $\mu_k \neq 0$ .

$\square$

With the proposition presented above, the following examples may be better understood.

**Example (1)** If  $A = K[a_1, \dots, a_n]$  is an  $\mathbb{N}$ -graded solvable polynomial algebra with respect to a positive-degree function  $d(\ )$ , i.e.,  $A = \bigoplus_{p \in \mathbb{N}} A_p$  with the degree- $p$  homogeneous part  $A_p = K\text{-span}\{a^\alpha \in \mathcal{B} \mid d(a^\alpha) = p\}$  (see Section 4.1 of Chapter 4), then, with respect to the same positive-degree function  $d(\ )$  on  $A$ ,  $A$  is turned into an  $\mathbb{N}$ -filtered solvable polynomial algebra with the  $\mathbb{N}$ -filtration  $FA = \{F_p A\}_{p \in \mathbb{N}}$  where each  $F_p A = \bigoplus_{q \leq p} A_q$ .

**Example (2)** Let  $A = K[a_1, \dots, a_n]$  be a solvable polynomial algebra with the admissible system  $(\mathcal{B}, \prec_{gr})$ , where  $\prec_{gr}$  is a graded monomial ordering on  $\mathcal{B}$  with respect to a given positive-degree function  $d(\ )$  on  $A$  (see the definition of  $\prec_{gr}$  given in Section 1.1 of Chapter 1). Then by referring to (Definition 1.1.3 of Chapter 1) and the above proposition, one easily sees that  $A$  is an  $\mathbb{N}$ -filtered solvable polynomial algebra with respect to the same  $d(\ )$ . In the case where  $\prec_{gr}$  respects  $d(a_i) = 1$  for  $1 \leq i \leq n$ ,

(Definition 1.1.3 of Chapter 1) entails that the generators of  $A$  satisfy

$$a_j a_i = \lambda_{ji} a_i a_j + \sum \lambda_{k\ell}^{ji} a_k a_\ell + \sum \lambda_t^{ji} a_t + \mu_{ji},$$

where  $1 \leq i < j \leq n$ ,  $\lambda_{ji} \in K^*$ ,  $\lambda_{k\ell}^{ji}, \lambda_t^{ji}, \mu_{ji} \in K$ .

In [Li1] such  $\mathbb{N}$ -filtered solvable polynomial algebras are referred to as *quadric solvable polynomial algebras* which include numerous significant algebras such as Weyl algebras and enveloping algebras of finite dimensional Lie algebras. One is referred to [Li1] for some detailed study on quadric solvable polynomial algebras.

The next example provides  $\mathbb{N}$ -filtered solvable polynomial algebras in which some generators may have degree  $\geq 2$ .

**Example (3)** Consider the solvable polynomial algebra  $K$ -algebra  $A = K[a_1, a_2, a_3]$  constructed in (Example (1) of Section 1.4, Chapter 1). Then, one checks that  $A$  is turned into an  $\mathbb{N}$ -filtered solvable polynomial algebra with respect to the lexicographic ordering  $a_3 \prec_{lex} a_2 \prec_{lex} a_1$  and the degree function  $d(\ )$  such that  $d(a_1) = 2$ ,  $d(a_2) = 1$ ,  $d(a_3) = 4$ . Moreover, one may also check that with respect to the same degree function  $d(\ )$ , the graded lexicographic ordering  $a_3 \prec_{grlex} a_2 \prec_{grlex} a_1$  is another choice to make  $A$  into an  $\mathbb{N}$ -filtered solvable polynomial algebra.

Let  $A$  be an  $\mathbb{N}$ -filtered solvable polynomial algebra with respect to a given positive-degree function  $d(\ )$ , and let  $FA = \{F_p A\}_{p \in \mathbb{N}}$  be the  $\mathbb{N}$ -filtration of  $A$  determined by  $d(\ )$ . Then  $A$  has the associated  $\mathbb{N}$ -graded  $K$ -algebra  $G(A) = \oplus_{p \in \mathbb{N}} G(A)_p$  with  $G(A)_0 = F_0 A = K$  and  $G(A)_p = F_p A / F_{p-1} A$  for  $p \geq 1$ , where for  $\bar{f} = f + F_{p-1} A \in G(A)_p$ ,  $\bar{g} = g + F_{q-1} A$ , the multiplication is given by  $\bar{f}\bar{g} = fg + F_{p+q-1} A \in G(A)_{p+q}$ . Another  $\mathbb{N}$ -graded  $K$ -algebra determined by  $FA$  is the Rees algebra  $\tilde{A}$  of  $A$ , which is defined as  $\tilde{A} = \oplus_{p \in \mathbb{N}} \tilde{A}_p$  with  $\tilde{A}_p = F_p A$ , where the multiplication of  $\tilde{A}$  is induced by  $F_p A F_q A \subseteq F_{p+q} A$ ,  $p, q \in \mathbb{N}$ . For convenience, we fix the following notations once for all:

- If  $h \in G(A)_p$  and  $h \neq 0$ , then we write  $d_{gr}(h)$  for the gr-degree of  $h$  as a homogeneous element of  $G(A)$ , i.e.,  $d_{gr}(h) = p$ .
- If  $H \in \tilde{A}_p$  and  $H \neq 0$ , then we write  $d_{gr}(H)$  for the gr-degree of  $H$  as a homogeneous element of  $\tilde{A}$ , i.e.,  $d_{gr}(H) = p$ .

Concerning the  $\mathbb{N}$ -graded structure of  $G(A)$ , if  $f \in A$  with  $d_{fil}(f) = p$ , then by Lemma 5.1.2, the coset  $f + F_{p-1} A$  represented by  $f$  in  $G(A)_p$  is a

nonzero homogeneous element of degree  $p$ . If we denote this homogeneous element by  $\sigma(f)$  (in the literature it is referred to as the principal symbol of  $f$ ), then  $d_{\text{fil}}(f) = p = d_{\text{gr}}(\sigma(f))$ . However, considering the Rees algebra  $\tilde{A}$  of  $A$ , we note that a nonzero  $f \in F_q A$  represents a homogeneous element of degree  $q$  in  $\tilde{A}_q$  on one hand, and on the other hand it represents a homogeneous element of degree  $q_1$  in  $\tilde{A}_{q_1}$ , where  $q_1 = d_{\text{fil}}(f) \leq q$ . So, for a nonzero  $f \in F_p A$ , we denote the corresponding homogeneous element of degree  $p$  in  $\tilde{A}_p$  by  $h_p(f)$ , while we use  $\tilde{f}$  to denote the homogeneous element represented by  $f$  in  $\tilde{A}_{p_1}$  with  $p_1 = d_{\text{fil}}(f) \leq p$ . Thus,  $d_{\text{gr}}(\tilde{f}) = d_{\text{fil}}(f)$ , and we see that  $h_p(f) = \tilde{f}$  if and only if  $d_{\text{fil}}(f) = p$ .

Furthermore, if we write  $Z$  for the homogeneous element of degree 1 in  $\tilde{A}_1$  represented by the multiplicative identity element 1, then  $Z$  is a central regular element of  $\tilde{A}$ , i.e.,  $Z$  is not a divisor of zero and is contained in the center of  $\tilde{A}$ . Bringing this homogeneous element  $Z$  into play, the homogeneous elements of  $\tilde{A}$  are featured as follows:

- If  $f \in A$  with  $d_{\text{fil}}(f) = p_1$  then for all  $p \geq p_1$ ,  $h_p(f) = Z^{p-p_1} \tilde{f}$ . In other words, if  $H \in \tilde{A}_p$  is a nonzero homogeneous element of degree  $p$ , then there is some  $f \in F_{p_1} A$  such that  $H = Z^{p-d_{\text{fil}}(f)} \tilde{f} = \tilde{f} + (Z^{p-d_{\text{fil}}(f)} - 1) \tilde{f}$ .

It follows that by sending  $H$  to  $f + F_{p-1} A$  and sending  $H$  to  $f$  respectively,  $G(A) \cong \tilde{A}/\langle Z \rangle$  as  $\mathbb{N}$ -graded  $K$ -algebras and  $A \cong \tilde{A}/\langle 1 - Z \rangle$  as  $K$ -algebras (cf. [LVO]).

Since a solvable polynomial algebra  $A$  is necessarily a domain (Proposition 1.1.4, Chapter 1), we summarize two useful properties concerning the multiplication of  $G(A)$  and  $\tilde{A}$  respectively into the following lemma.

**5.1.4. Lemma** With notation as before, let  $f, g$  be nonzero elements of  $A$  with  $d_{\text{fil}}(f) = p_1$ ,  $d_{\text{fil}}(g) = p_2$ . Then

- (i)  $\sigma(f)\sigma(g) = \sigma(fg)$ ;
- (ii)  $\tilde{f}\tilde{g} = fg$ . If  $p_1 + p_2 \leq p$ , then  $h_p(fg) = Z^{p-p_1-p_2} \tilde{f}\tilde{g}$ .

**Proof** Exercise. □

The results given in the next theorem, which are analogues of those concerning quadric solvable polynomial algebras in ([LW], Section 3; [Li1], CH.IV), may be derived in a similar way as in loc. cit. (With the preparation made above, one is also invited to give the detailed proof as an



exercise).

**5.1.5. Theorem** Let  $A = K[a_1, \dots, a_n]$  be a solvable polynomial algebra with the admissible system  $(\mathcal{B}, \prec_{gr})$ , where  $\prec_{gr}$  is a graded monomial ordering on  $\mathcal{B}$  with respect to a given positive-degree function  $d(\cdot)$  on  $A$ , thereby  $A$  is an  $\mathbb{N}$ -filtered solvable polynomial algebra with respect to the same  $d(\cdot)$  by the foregoing Example (2), and let  $FA = \{F_p A\}_{p \in \mathbb{N}}$  be the corresponding  $\mathbb{N}$ -filtration of  $A$ . Considering the associated graded algebra  $G(A)$  as well as the Rees algebra  $\tilde{A}$  of  $A$ , the following statements hold.

(i)  $G(A) = K[\sigma(a_1), \dots, \sigma(a_n)]$ ,  $G(A)$  has the PBW  $K$ -basis

$$\sigma(\mathcal{B}) = \{\sigma(a)^\alpha = \sigma(a_1)^{\alpha_1} \cdots \sigma(a_n)^{\alpha_n} \mid \alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n\},$$

and, by referring to (Definition 1.1.3, Chapter 1), for  $\sigma(a)^\alpha, \sigma(a)^\beta \in \sigma(\mathcal{B})$  such that  $a^\alpha a^\beta = \lambda_{\alpha, \beta} a^{\alpha+\beta} + f_{\alpha, \beta}$ , where  $\lambda_{\alpha, \beta} \in K^*$ , if  $f_{\alpha, \beta} = 0$  then

$$\sigma(a)^\alpha \sigma(a)^\beta = \lambda_{\alpha, \beta} \sigma(a)^{\alpha+\beta}, \text{ where } \sigma(a)^{\alpha+\beta} = \sigma(a_1)^{\alpha_1+\beta_1} \cdots \sigma(a_n)^{\alpha_n+\beta_n};$$

and in the case where  $f_{\alpha, \beta} = \sum_j \mu_j^{\alpha, \beta} a^{\alpha(j)} \neq 0$  with  $\mu_j^{\alpha, \beta} \in K$ ,

$$\sigma(a)^\alpha \sigma(a)^\beta = \lambda_{\alpha, \beta} \sigma(a)^{\alpha+\beta} + \sum_{d(a^{\alpha(k)})=d(a^{\alpha+\beta})} \mu_j^{\alpha, \beta} \sigma(a)^{\alpha(k)}.$$

Moreover, the ordering  $\prec_{G(A)}$  defined on  $\sigma(\mathcal{B})$  subject to the rule:

$$\sigma(a)^\alpha \prec_{G(A)} \sigma(a)^\beta \iff a^\alpha \prec_{gr} a^\beta, \quad a^\alpha, a^\beta \in \mathcal{B},$$

is a graded monomial ordering with respect to the positive-degree function  $d(\cdot)$  on  $G(A)$  such that  $d(\sigma(a_i)) = d(a_i)$  for  $1 \leq i \leq n$ , that turns  $G(A)$  into an  $\mathbb{N}$ -graded solvable polynomial algebra.

(ii)  $\tilde{A} = K[\tilde{a}_1, \dots, \tilde{a}_n, Z]$  where  $Z$  is the central regular element of degree 1 in  $\tilde{A}_1$  represented by 1,  $\tilde{A}$  has the PBW  $K$ -basis

$$\tilde{\mathcal{B}} = \{\tilde{a}^\alpha Z^m = \tilde{a}_1^{\alpha_1} \cdots \tilde{a}_n^{\alpha_n} Z^m \mid \alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n, m \in \mathbb{N}\},$$

and, by referring to (Definition 1.1.3, Chapter 1), for  $\tilde{a}^\alpha Z^s, \tilde{a}^\beta Z^t \in \tilde{\mathcal{B}}$  such that  $a^\alpha a^\beta = \lambda_{\alpha, \beta} a^{\alpha+\beta} + f_{\alpha, \beta}$ , where  $\lambda_{\alpha, \beta} \in K^*$ , if  $f_{\alpha, \beta} = 0$  then

$$\tilde{a}^\alpha Z^s \cdot \tilde{a}^\beta Z^t = \lambda_{\alpha, \beta} \tilde{a}^{\alpha+\beta} Z^{s+t}, \text{ where } \tilde{a}^{\alpha+\beta} = \tilde{a}_1^{\alpha_1+\beta_1} \cdots \tilde{a}_n^{\alpha_n+\beta_n};$$

and in the case where  $f_{\alpha,\beta} = \sum_j \mu_j^{\alpha,\beta} a^{\alpha(j)} \neq 0$  with  $\mu_j^{\alpha,\beta} \in K$ ,

$$\begin{aligned} \tilde{a}^\alpha Z^s \cdot \tilde{a}^\beta Z^t &= \lambda_{\alpha,\beta} \tilde{a}^{\alpha+\beta} Z^{s+t} + \sum_j \mu_j^{\alpha,\beta} \tilde{a}^{\alpha(j)} Z^{q-m_j}, \\ &\text{where } q = d(a^{\alpha+\beta}) + s + t, \quad m_j = d(a^{\alpha(j)}). \end{aligned}$$

Moreover, the ordering  $\prec_{\tilde{A}}$  defined on  $\tilde{\mathcal{B}}$  subject to the rule:

$$\tilde{a}^\alpha Z^s \prec_{\tilde{A}} \tilde{a}^\beta Z^t \iff a^\alpha \prec_{gr} a^\beta, \text{ or } a^\alpha = a^\beta \text{ and } s < t, \quad a^\alpha, a^\beta \in \mathcal{B},$$

is a monomial ordering on  $\tilde{\mathcal{B}}$  (which is not necessarily a graded monomial ordering), that turns  $\tilde{A}$  into an  $\mathbb{N}$ -graded solvable polynomial algebra with respect to the positive-degree function  $d(\cdot)$  on  $\tilde{A}$  such that  $d(Z) = 1$  and  $d(\tilde{a}_i) = d(a_i)$  for  $1 \leq i \leq n$ . □

The corollary presented below will be very often used in discussing left Gröbner bases and standard bases for submodules of filtered free  $A$ -modules and their associated graded free  $G(A)$ -modules as well as the graded free  $\tilde{A}$ -modules (Section 5.3, Section 5.4).

**5.1.6. Corollary** With the assumption and notations as in Theorem 3.1.4, if  $f = \lambda a^\alpha + \sum_j \mu_j a^{\alpha(j)}$  with  $d(f) = p$  and  $\mathbf{LM}(f) = a^\alpha$ , then  $p = d_{\text{fil}}(f) = d_{\text{gr}}(\sigma(f)) = d_{\text{gr}}(\tilde{f})$ , and

$$\begin{aligned} \sigma(f) &= \lambda \sigma(a)^\alpha + \sum_{d(a^{\alpha(j_k)})=p} \mu_{j_k} \sigma(a)^{\alpha(j_k)}; \\ \mathbf{LM}(\sigma(f)) &= \sigma(a)^\alpha = \sigma(\mathbf{LM}(f)); \\ \tilde{f} &= \lambda \tilde{a}^\alpha + \sum_j \mu_j \tilde{a}^{\alpha(j)} Z^{p-d(a^{\alpha(j)})}; \\ \mathbf{LM}(\tilde{f}) &= \tilde{a}^\alpha = \mathbf{LM}(f), \end{aligned}$$

where  $\mathbf{LM}(f)$ ,  $\mathbf{LM}(\sigma(f))$  and  $\mathbf{LM}(\tilde{f})$  are taken with respect to  $\prec_{gr}$ ,  $\prec_{G(A)}$  and  $\prec_{\tilde{A}}$  respectively.

**Proof** By referring to Lemma 5.1.2 and Lemma 5.1.4, this is a straightforward exercise. □

## 5.2. $\mathbb{N}$ -Filtered Free Modules

Let  $A = K[a_1, \dots, a_n]$  be an  $\mathbb{N}$ -filtered solvable polynomial algebra with the filtration  $FA = \{F_p A\}_{p \in \mathbb{N}}$  determined by a positive-degree function

$d(\cdot)$  on  $A$ , and let  $(\mathcal{B}, \prec)$  be a fixed admissible system of  $A$ . Consider a free  $A$ -module  $L = \oplus_{i=1}^s Ae_i$  with the  $A$ -basis  $\{e_1, \dots, e_s\}$ . Then  $L$  has the  $K$ -basis  $\mathcal{B}(e) = \{a^\alpha e_i \mid a^\alpha \in \mathcal{B}, 1 \leq i \leq s\}$ . If  $\{b_1, \dots, b_s\}$  is an *arbitrarily* fixed subset of  $\mathbb{N}$ , then, with  $FL = \{F_q L\}_{q \in \mathbb{N}}$  defined by putting

$$F_q L = \{0\} \text{ if } q < \min\{b_1, \dots, b_s\}; \text{ otherwise } F_q L = \sum_{i=1}^s \left( \sum_{p_i + b_i \leq q} F_{p_i} A \right) e_i,$$

or alternatively, for  $q \geq \min\{b_1, \dots, b_s\}$ ,

$$F_q L = K\text{-span}\{a^\alpha e_i \in \mathcal{B}(e) \mid d(a^\alpha) + b_i \leq q\},$$

$L$  forms an  $\mathbb{N}$ -filtered free  $A$ -module with respect to the  $\mathbb{N}$ -filtered structure of  $A$ , that is, every  $F_q L$  is a  $K$ -subspace of  $L$ ,  $F_q L \subseteq F_{q+1} L$  for all  $q \in \mathbb{N}$ ,  $L = \cup_{q \in \mathbb{N}} F_q L$ ,  $F_p A F_q L \subseteq F_{p+q} L$  for all  $p, q \in \mathbb{N}$ , and for each  $i = 1, \dots, s$ ,

$$e_i \in F_0 L \text{ if } b_i = 0; \text{ otherwise } e_i \in F_{b_i} L - F_{b_i-1} L.$$

**Convention.** Let  $A$  be an  $\mathbb{N}$ -filtered solvable polynomial algebra with respect to a positive-degree function  $d(\cdot)$ . Unless otherwise stated, from now on in the subsequent sections if we say that  $L = \oplus_{i=1}^s Ae_i$  is a filtered free  $A$ -module with the filtration  $FL = \{F_q L\}_{q \in \mathbb{N}}$ , then  $FL$  is always meant the type as constructed above.

Let  $L = \oplus_{i=1}^s Ae_i$  be a filtered free  $A$ -module with the filtration  $FL = \{F_q L\}_{q \in \mathbb{N}}$ , which is constructed with respect to a given subset  $\{b_1, \dots, b_s\} \subset \mathbb{N}$ . Then  $FL$  is *separated* in the sense that if  $\xi$  is a *nonzero* element of  $L$ , then either  $\xi \in F_0 L$  or  $\xi \in F_q L - F_{q-1} L$  for some  $q > 0$ . Thus, to make the discussion on  $FL$  consistent with that on  $FA$  in Section 5.1, we define the *filtered-degree* (abbreviated to *fil-degree*) of a nonzero  $\xi \in L$ , denoted  $d_{\text{fil}}(\xi)$ , as follows

$$d_{\text{fil}}(\xi) = \begin{cases} 0, & \text{if } \xi \in F_0 L, \\ q, & \text{if } \xi \in F_q L - F_{q-1} L \text{ for some } q > 0. \end{cases}$$

For instance, we have  $d_{\text{fil}}(e_i) = b_i$ ,  $1 \leq i \leq s$ . Comparing with Lemma 5.1.2 we first note the following

**5.2.1. Lemma** Let  $\xi \in L - \{0\}$ . Then  $d_{\text{fil}}(\xi) = q$  if and only if  $\xi = \sum_{i,j} \lambda_{ij} a^{\alpha(i_j)} e_j$ , where  $\lambda_{ij} \in K^*$  and  $a^{\alpha(i_j)} \in \mathcal{B}$  with  $\alpha(i_j) = (\alpha_{i_{j1}}, \dots, \alpha_{i_{jn}}) \in \mathbb{N}^n$ , in which some monomial  $a^{\alpha(i_j)} e_j$  satisfy  $d(a^{\alpha(i_j)}) + b_j = q$ .

**Proof** Exercise. □

Let  $L = \bigoplus_{i=1}^s A e_i$  be a filtered free  $A$ -module with the filtration  $FL = \{F_q L\}_{q \in \mathbb{N}}$  such that  $d_{\text{fil}}(e_i) = b_i$ ,  $1 \leq i \leq s$ . Considering the associated  $\mathbb{N}$ -graded algebra  $G(A)$  of  $A$ , the filtered free  $A$  module  $L$  has the associated  $\mathbb{N}$ -graded  $G(A)$ -module  $G(L) = \bigoplus_{q \in \mathbb{N}} G(L)_q$  with  $G(L)_q = F_q L / F_{q-1} L$ , where for  $\bar{f} = f + F_{p-1} A \in G(A)_p$ ,  $\bar{\xi} = \xi + F_{q-1} L \in G(L)_q$ , the module action is given by  $\bar{f} \cdot \bar{\xi} = f\xi + F_{p+q-1} L \in G(L)_{p+q}$ . As with homogeneous elements in  $G(A)$ , if  $h \in G(L)_q$  and  $h \neq 0$ , then we write  $d_{\text{gr}}(h)$  for the degree of  $h$  as a homogeneous element of  $G(L)$ , i.e.,  $d_{\text{gr}}(h) = q$ . If  $\xi \in L$  with  $d_{\text{fil}}(\xi) = q$ , then the coset  $\xi + F_{q-1} L$  represented by  $\xi$  in  $G(L)_q$  is a nonzero homogeneous element of degree  $q$ , and if we denote this homogeneous element by  $\sigma(\xi)$  (in the literature it is referred to as the principal symbol of  $\xi$ ) then  $d_{\text{gr}}(\sigma(\xi)) = q = d_{\text{fil}}(\xi)$ .

Furthermore, considering the Rees algebra  $\tilde{A}$  of  $A$ , the filtration  $FL = \{F_q L\}_{q \in \mathbb{N}}$  of  $L$  also defines the Rees module  $\tilde{L}$  of  $L$ , which is the  $\mathbb{N}$ -graded  $\tilde{A}$ -module  $\tilde{L} = \bigoplus_{q \in \mathbb{N}} \tilde{L}_q$ , where  $\tilde{L}_q = F_q L$  and the module action is induced by  $F_p A F_q L \subseteq F_{p+q} L$ . As with homogeneous elements in  $\tilde{A}$ , if  $H \in \tilde{L}_q$  and  $H \neq 0$ , then we write  $d_{\text{gr}}(H)$  for the degree of  $H$  as a homogeneous element of  $\tilde{L}$ , i.e.,  $d_{\text{gr}}(H) = q$ . Note that any nonzero  $\xi \in F_q L$  represents a homogeneous element of degree  $q$  in  $\tilde{L}_q$  on one hand, and on the other hand it represents a homogeneous element of degree  $q_1$  in  $\tilde{L}_{q_1}$ , where  $q_1 = d_{\text{fil}}(\xi) \leq q$ . So, for a nonzero  $\xi \in F_q L$  we denote the corresponding homogeneous element of degree  $q$  in  $\tilde{L}_q$  by  $h_q(\xi)$ , while we use  $\tilde{\xi}$  to denote the homogeneous element represented by  $\xi$  in  $\tilde{L}_{q_1}$  with  $q_1 = d_{\text{fil}}(\xi) \leq q$ . Thus,  $d_{\text{gr}}(\tilde{\xi}) = d_{\text{fil}}(\xi)$ , and we see that  $h_q(\xi) = \tilde{\xi}$  if and only if  $d_{\text{fil}}(\xi) = q$ .

We also note that if  $Z$  denotes the homogeneous element of degree 1 in  $\tilde{A}_1$  represented by the multiplicative identity element 1, then, similar to the discussion given in the last section, one checks that there are  $A$ -module isomorphism  $L \cong \tilde{L}/(1 - Z)\tilde{L}$  and graded  $G(A)$ -module isomorphism  $G(L) \cong \tilde{L}/Z\tilde{L}$ .

For  $f \in A$ ,  $\xi \in L$ , the next lemma records several convenient facts about  $d_{\text{fil}}(f\xi)$ ,  $\sigma(f\xi)$ ,  $h_\ell(f\xi)$  and  $\tilde{f\xi}$ , respectively.

**5.2.2. Lemma** With notation as above, the following statements hold.

- (i)  $d_{\text{fil}}(f\xi) = d(f) + d_{\text{fil}}(\xi)$  holds for all nonzero  $f \in A$  and nonzero  $\xi \in L$ .
- (ii)  $\sigma(f)\sigma(\xi) = \sigma(f\xi)$  holds for all nonzero  $f \in A$  and nonzero  $\xi \in L$ .
- (iii) If  $\xi \in L$  with  $d_{\text{fil}}(\xi) = q \leq \ell$ , then  $h_\ell(\xi) = Z^{\ell-q}\tilde{\xi}$ . Furthermore, let  $f \in A$  with  $d_{\text{fil}}(f) = p$ ,  $\xi \in L$  with  $d_{\text{fil}}(\xi) = q$ . Then  $\widetilde{f\xi} = \widetilde{f}\tilde{\xi}$ ; if  $p+q \leq \ell$ , then  $h_\ell(f\xi) = Z^{\ell-p-q}\widetilde{f\xi}$ .

**Proof** Since  $A$  is a solvable polynomial algebra,  $G(A)$  and  $\tilde{A}$  are  $\mathbb{N}$ -graded solvable polynomial algebras by Theorem 5.1.5, thereby they are necessarily domains (Proposition 1.1.4(ii), Chapter 1). Noticing the definition of  $d_{\text{fil}}(f)$  and  $d_{\text{fil}}(\xi)$ , by Lemma 5.2.1, the verification of (i) – (iii) are then straightforward.  $\square$

**5.2.3. Proposition** With notation as fixed before, let  $L = \bigoplus_{i=1}^s Ae_i$  be a filtered free  $A$ -module with the filtration  $FL = \{F_q L\}_{q \in \mathbb{N}}$  such that  $d_{\text{fil}}(e_i) = b_i$ ,  $1 \leq i \leq s$ . The following two statements hold.

- (i)  $G(L)$  is an  $\mathbb{N}$ -graded free  $G(A)$ -module with the homogeneous  $G(A)$ -basis  $\{\sigma(e_1), \dots, \sigma(e_s)\}$ , that is,  $G(L) = \bigoplus_{i=1}^s G(A)\sigma(e_i) = \bigoplus_{q \in \mathbb{N}} G(L)_q$  with

$$G(L)_q = \sum_{p_i + b_i = q} G(A)_{p_i} \sigma(e_i) \quad q \in \mathbb{N}.$$

Moreover,  $\sigma(\mathcal{B}(e)) = \{\sigma(a^\alpha e_i) = \sigma(a)^\alpha \sigma(e_i) \mid a^\alpha e_i \in \mathcal{B}(e)\}$  forms a  $K$ -basis for  $G(L)$ .

- (ii)  $\tilde{L}$  is an  $\mathbb{N}$ -graded free  $\tilde{A}$ -module with the homogeneous  $\tilde{A}$ -basis  $\{\tilde{e}_1, \dots, \tilde{e}_s\}$ , that is,  $\tilde{L} = \bigoplus_{i=1}^s \tilde{A}\tilde{e}_i = \bigoplus_{q \in \mathbb{N}} \tilde{L}_q$  with

$$\tilde{L}_q = \sum_{p_i + b_i = q} \tilde{A}_{p_i} \tilde{e}_i, \quad q \in \mathbb{N}.$$

Moreover,  $\widetilde{\mathcal{B}(e)} = \{\tilde{a}^\alpha Z^m \tilde{e}_i \mid \tilde{a}^\alpha Z^m \in \tilde{\mathcal{B}}, 1 \leq i \leq s\}$  forms a  $K$ -basis for  $\tilde{L}$ , where  $\tilde{\mathcal{B}}$  is the PBW  $K$ -basis of  $\tilde{A}$  determined in Theorem 5.1.5(ii).

**Proof** Since  $d_{\text{fil}}(e_i) = b_i$ ,  $1 \leq i \leq s$ , if  $\xi = \sum_{i=1}^s f_i e_i \in F_q L = \sum_{i=1}^s \left( \sum_{p_i + b_i \leq q} F_{p_i} A \right) e_i$ , then  $d_{\text{fil}}(\xi) \leq q$ . By Lemma 5.2.2,

$$\sigma(\xi) = \sum_{d(f_i) + b_i = q} \sigma(f_i) \sigma(e_i) \in \sum_{i=1}^s G(A)_{q-b_i} \sigma(e_i)$$

$$h_q(\xi) = \sum_{i=1}^s Z^{q-d(f_i)-b_i} \tilde{f}_i \tilde{e}_i \in \sum_{i=1}^s \tilde{A}_{q-b_i} \tilde{e}_i.$$

This shows that  $\{\sigma(e_1), \dots, \sigma(e_s)\}$  and  $\{\tilde{e}_1, \dots, \tilde{e}_s\}$  generate the  $G(A)$ -module  $G(L)$  and the  $\tilde{A}$ -module  $\tilde{L}$ , respectively. Next, since each  $\sigma(e_i)$  is

a homogeneous element of degree  $b_i$ , if a degree- $q$  homogeneous element  $\sum_{i=1}^s \sigma(f_i)\sigma(e_i) = 0$ , where  $f_i \in A$ ,  $d_{\text{fil}}(f_i) + b_i = q$ ,  $1 \leq i \leq s$ , then  $\sum_{i=1}^s f_i e_i \in F_{q-1}L$  and hence each  $f_i \in F_{q-1-b_i}A$  by Lemma 5.2.1, a contradiction. It follows that  $\{\sigma(e_1), \dots, \sigma(e_s)\}$  is linearly independent over  $G(A)$ . Concerning the linear independence of  $\{\tilde{e}_1, \dots, \tilde{e}_s\}$  over  $\tilde{A}$ , since each  $\tilde{e}_i$  is a homogeneous element of degree  $b_i$ , if a degree- $q$  homogeneous element  $\sum_{i=1}^s h_{p_i}(f_i)\tilde{e}_i = 0$ , where  $f_i \in F_{p_i}A$  and  $p_i + b_i = q$ ,  $1 \leq i \leq s$ , then  $\sum_{i=1}^s f_i e_i = 0$  in  $F_q L$  and consequently all  $f_i = 0$ , thereby  $h_{p_i}(f_i) = 0$  as desired. Finally, if  $\xi \in F_q L$  with  $d_{\text{fil}}(\xi) = q$ , then by Lemma 5.2.1,  $\xi = \sum_{i,j} \lambda_{ij} a^{\alpha(i_j)} e_j$  with  $\lambda_{ij} \in K^*$  and  $d(a^{\alpha(i_j)}) + b_j = \ell_{ij} \leq q$ . It follows from Lemma 5.2.2 that

$$\begin{aligned} \sigma(\xi) &= \sum_{\ell_{ik}=q} \lambda_{ik} \sigma(a)^{\alpha(i_k)} \sigma(e_k), \\ \tilde{\xi} &= \sum_{i,j} \lambda_{ij} Z^{q-\ell_{ij}} \tilde{a}^{\alpha(i_j)} \tilde{e}_j. \end{aligned}$$

Therefore, a further application of Lemma 5.2.1 and Lemma 5.2.2 shows that  $\sigma(\mathcal{B}(e))$  and  $\widetilde{\mathcal{B}(e)}$  are  $K$ -bases for  $G(L)$  and  $\tilde{L}$  respectively.

### 5.3. Filtered-Graded Transfer of Gröbner Bases for Modules

Throughout this section, we let  $A = K[a_1, \dots, a_n]$  be a solvable polynomial algebra with the admissible system  $(\mathcal{B}, \prec_{gr})$ , where  $\mathcal{B} = \{a^\alpha = a_1^{\alpha_1} \cdots a_n^{\alpha_n} \mid \alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n\}$  is the PBW  $K$ -basis of  $A$  and  $\prec_{gr}$  is a graded monomial ordering with respect to some given positive-degree function  $d(\cdot)$  on  $A$  (see Section 1). Thereby  $A$  is turned into an  $\mathbb{N}$ -filtered solvable polynomial algebra with the filtration  $FA = \{F_p A\}_{p \in \mathbb{N}}$  constructed with respect to the same  $d(\cdot)$  (see Example (2) of Section 5.1). In order to compute minimal standard bases by employing both inhomogeneous and homogenous left Gröbner bases in the subsequent Section 5.5, our aim of the current section is to establish the relations between left Gröbner bases in a filtered free (left)  $A$ -module  $L$  and homogeneous left Gröbner bases in  $G(L)$  as well as homogeneous left Gröbner bases in  $\tilde{L}$ , which are just module theory analogues of the results on filtered-graded transfer of Gröbner bases for left ideals given in ([LW], [Li1]). All notions, notations and conventions introduced in previous sections are maintained.

Let  $L = \bigoplus_{i=1}^s A e_i$  be a filtered free  $A$ -module with the filtration  $FL =$

$\{F_q L\}_{q \in \mathbb{N}}$  such that  $d_{\text{fil}}(e_i) = b_i$ ,  $1 \leq i \leq s$ . Bearing in mind Lemma 5.2.1, we say that a left monomial ordering on  $\mathcal{B}(e)$  is a *graded left monomial ordering*, denoted by  $\prec_{e\text{-gr}}$ , if for  $a^\alpha e_i, a^\beta e_j \in \mathcal{B}(e)$ ,

$$a^\alpha e_i \prec_{e\text{-gr}} a^\beta e_j \text{ implies } d_{\text{fil}}(a^\alpha e_i) = d(a^\alpha) + b_i \leq d(a^\beta) + b_j = d_{\text{fil}}(a^\beta e_j).$$

For instance, with respect to the given graded monomial ordering  $\prec_{gr}$  on  $\mathcal{B}$  and the  $\mathbb{N}$ -filtration  $FA$  of  $A$ , if  $\{f_1, \dots, f_s\} \subset A$  is a finite subset such that  $d(f_i) = b_i = d_{\text{fil}}(e_i)$ ,  $1 \leq i \leq s$ , then it is straightforward to check that the Schreyer ordering (see Section 2.1 of Chapter 2) induced by  $\{f_1, \dots, f_s\}$  subject to the rule: for  $a^\alpha e_i, a^\beta e_j \in \mathcal{B}(e)$ ,

$$a^\alpha e_i \prec_{s\text{-gr}} a^\beta e_j \iff \begin{cases} \mathbf{LM}(a^\alpha f_i) \prec_{gr} \mathbf{LM}(a^\beta f_j), \\ \text{or} \\ \mathbf{LM}(a^\alpha f_i) = \mathbf{LM}(a^\beta f_j) \text{ and } i < j. \end{cases}$$

is a graded left monomial ordering on  $\mathcal{B}(e)$ .

More generally, let  $\{\xi_1, \dots, \xi_m\} \subset L$  be a finite subset, where  $d_{\text{fil}}(\xi_i) = q_i$ ,  $1 \leq i \leq m$ , and let  $L_1 = \oplus_{i=1}^m A\xi_i$  be the filtered free  $A$ -module with the filtration  $FL_1 = \{F_q L_1\}_{q \in \mathbb{N}}$  such that  $d_{\text{fil}}(\varepsilon_i) = q_i$ ,  $1 \leq i \leq m$ . Then, given *any* graded left monomial ordering  $\prec_{e\text{-gr}}$  on  $\mathcal{B}(e)$ , the Schreyer ordering  $\prec_{s\text{-gr}}$  defined on the  $K$ -basis  $\mathcal{B}(\varepsilon) = \{a^\alpha \varepsilon_i \mid a^\alpha \in \mathcal{B}, 1 \leq i \leq m\}$  of  $L_1$  subject to the rule: for  $a^\alpha \varepsilon_i, a^\beta \varepsilon_j \in \mathcal{B}(\varepsilon)$ ,

$$a^\alpha \varepsilon_i \prec_{s\text{-gr}} a^\beta \varepsilon_j \iff \begin{cases} \mathbf{LM}(a^\alpha \xi_i) \prec_{e\text{-gr}} \mathbf{LM}(a^\beta \xi_j), \\ \text{or} \\ \mathbf{LM}(a^\alpha \xi_i) = \mathbf{LM}(a^\beta \xi_j) \text{ and } i < j, \end{cases}$$

is a graded left monomial ordering on  $\mathcal{B}(\varepsilon)$ .

Comparing with Lemma 5.1.2 and Lemma 5.2.1, the lemma given below reveals the intrinsic property of a graded left monomial ordering employed by a filtered free  $A$ -module.

**5.3.1. Lemma** Let  $L = \oplus_{i=1}^s A e_i$  be a filtered free  $A$ -module with the filtration  $FL = \{F_q L\}_{q \in \mathbb{N}}$  such that  $d_{\text{fil}}(e_i) = b_i$ ,  $1 \leq i \leq s$ , and let  $\prec_{e\text{-gr}}$  be a graded left monomial ordering on  $\mathcal{B}(e)$ . Then  $\prec_{e\text{-gr}}$  is compatible with the filtration  $FL$  of  $L$  in the sense that  $\xi \in F_q L - F_{q-1} L$ , i.e.  $d_{\text{fil}}(\xi) = q$ , if and only if  $\mathbf{LM}(\xi) = a^\alpha e_i$  with  $d_{\text{fil}}(a^\alpha e_i) = d(a^\alpha) + b_i = q$ .

**Proof** Let  $\xi = \sum_{i,j} \lambda_{ij} a^{\alpha(ij)} e_j \in F_q L - F_{q-1} L$ . Then by Lemma 5.2.1, there is some  $a^{\alpha(i_\ell)} e_\ell$  such that  $d(a^{\alpha(i_\ell)}) + b_\ell = q$ . If  $\mathbf{LM}(\xi) = a^{\alpha(i_t)} e_t$  with

respect to  $\prec_{e-gr}$ , then  $a^{\alpha(i_k)}e_k \prec_{e-gr} a^{\alpha(i_t)}e_t$  for all  $a^{\alpha(i_k)}e_k$  with  $k \neq t$ . If  $\ell = t$ , then  $d(a^{\alpha(i_t)}) + b_t = q$ ; otherwise, since  $\prec_{e-gr}$  is a graded left monomial ordering, we have  $d(a^{\alpha(i_k)}) + b_k \leq d(a^{\alpha(i_t)}) + b_t$ , in particular,  $q = d(a^{\alpha(i_\ell)}) + b_\ell \leq d(a^{\alpha(i_t)}) + b_t \leq q$ . Hence  $d_{\text{fil}}(a^{\alpha(i_t)}e_t) = d(a^{\alpha(i_t)}) + b_t = q$ , as desired.

Conversely, for  $\xi = \sum_{i,j} \lambda_{ij} a^{\alpha(i_j)}e_j \in L$ , if, with respect to  $\prec_{e-gr}$ ,  $\mathbf{LM}(\xi) = a^{\alpha(i_t)}e_t$  with  $d_{\text{fil}}(a^{\alpha(i_t)}e_t) = d(a^{\alpha(i_t)}) + b_t = q$ , then  $a^{\alpha(i_k)}e_k \prec_{e-gr} a^{\alpha(i_t)}e_t$  for all  $k \neq t$ . Since  $\prec_{e-gr}$  is a graded left monomial ordering, we have  $d(a^{\alpha(i_k)}) + b_k \leq d(a^{\alpha(i_t)}) + b_t = q$ . It follows from Lemma 5.2.1 that  $d_{\text{fil}}(\xi) = q$ , i.e.,  $\xi \in F_q L - F_{q-1} L$ .  $\square$

Let  $L = \bigoplus_{i=1}^s A e_i$  be a filtered free  $A$ -module with the filtration  $FL = \{F_q L\}_{q \in \mathbb{N}}$  such that  $d_{\text{fil}}(e_i) = b_i$ ,  $1 \leq i \leq s$ . Then, by Proposition 5.2.3 we know that the associated graded  $G(A)$ -module  $G(L)$  of  $L$  is an  $\mathbb{N}$ -graded free module, i.e.,  $G(L) = \bigoplus_{i=1}^s G(A) \sigma(e_i)$  with the homogeneous  $G(A)$ -basis  $\{\sigma(e_1), \dots, \sigma(e_s)\}$ , and that  $G(L)$  has the  $K$ -basis  $\sigma(\mathcal{B}(e)) = \{\sigma(a^\alpha e_i) = \sigma(a)^\alpha \sigma(e_i) \mid a^\alpha e_i \in \mathcal{B}(e)\}$ . Furthermore, let  $\prec_{e-gr}$  be a graded left monomial ordering on  $\mathcal{B}(e)$  as defined in the beginning of this section. Then we may define an ordering  $\prec_{\sigma(e)-gr}$  on  $\sigma(\mathcal{B}(e))$  subject to the rule:

$$\sigma(a)^\alpha \sigma(e_i) \prec_{\sigma(e)-gr} \sigma(a)^\beta \sigma(e_j) \iff a^\alpha e_i \prec_{e-gr} a^\beta e_j, \quad a^\alpha e_i, a^\beta e_j \in \mathcal{B}(e).$$

**5.3.2. Lemma** With the ordering  $\prec_{\sigma(e)-gr}$  defined above, the following statements hold.

- (i)  $\prec_{\sigma(e)-gr}$  is a graded left monomial ordering on  $\sigma(\mathcal{B}(e))$ .
- (ii) (Compare with Corollary 5.1.6.)  $\mathbf{LM}(\sigma(\xi)) = \sigma(\mathbf{LM}(\xi))$  holds for all nonzero  $\xi \in L$ , where the monomial orderings used for taking  $\mathbf{LM}(\sigma(\xi))$  and  $\mathbf{LM}(\xi)$  are  $\prec_{\sigma(e)-gr}$  and  $\prec_{e-gr}$  respectively.

**Proof** (i) Noticing that the given monomial ordering  $\prec_{gr}$  on  $A$  is a graded monomial ordering with respect to a positive-degree function  $d(\cdot)$  on  $A$ , it follows from Theorem 5.1.5(i) that  $G(A)$  is turned into an  $\mathbb{N}$ -graded solvable polynomial algebra by using the graded monomial ordering  $\prec_{G(A)}$  defined on  $\sigma(\mathcal{B})$  subject to the rule:  $\sigma(a)^\alpha \prec_{G(A)} \sigma(a)^\beta \iff a^\alpha \prec_{gr} a^\beta$ , where the positive-degree function on  $G(A)$  is given by  $d(\sigma(a_i)) = d(a_i)$ ,  $1 \leq i \leq n$ . Moreover, since  $\sigma(e_i)$  is a homogeneous element of degree  $b_i$  in  $G(L)$ ,  $1 \leq i \leq s$ , by Lemma 5.2.2, it is then straightforward to verify that  $\prec_{\sigma(e)-gr}$  is a graded left monomial ordering on  $\sigma(\mathcal{B}(e))$ .



(ii) Let  $\xi = \sum_{i,j} \lambda_{ij} a^{\alpha(i_j)} e_j$ , where  $\lambda_{ij} \in K^*$  and  $a^{\alpha(i_j)} \in \mathcal{B}$  with  $\alpha(i_j) = (\alpha_{i_{j1}}, \dots, \alpha_{i_{jn}}) \in \mathbb{N}^n$ . If  $d_{\text{fil}}(\xi) = q$ , i.e.,  $\xi \in F_q L - F_{q-1} L$ , then by Lemma 5.3.1,  $\mathbf{LM}(\xi) = a^{\alpha(i_t)} e_t$  for some  $t$  such that  $d_{\text{fil}}(a^{\alpha(i_t)} e_t) = d(a^{\alpha(i_t)}) + b_t = q$ . Since  $\prec_{e-gr}$  is a left graded monomial ordering on  $\mathcal{B}(e)$ , by Lemma 5.2.2 we have  $\sigma(\xi) = \lambda_{it} \sigma(a)^{\alpha(i_t)} \sigma(e_t) + \sum_{d(a^{\alpha(i_k)}) + b_k = q} \lambda_{ik} \sigma(a)^{\alpha(i_k)} \sigma(e_k)$ . It follows from the definition of  $\prec_{\sigma(e)-gr}$  that  $\mathbf{LM}(\sigma(\xi)) = \sigma(a)^{\alpha(i_t)} \sigma(e_t) = \sigma(\mathbf{LM}(\xi))$ , as desired.  $\square$

**5.3.3. Theorem** Let  $N$  be a submodule of the filtered free  $A$ -module  $L = \bigoplus_{i=1}^s A e_i$ , where  $L$  is equipped with the filtration  $FL = \{F_q L\}_{q \in \mathbb{N}}$  such that  $d_{\text{fil}}(e_i) = b_i$ ,  $1 \leq i \leq s$ , and let  $\prec_{e-gr}$  be a graded left monomial ordering on  $\mathcal{B}(e)$ . For a subset  $\mathcal{G} = \{g_1, \dots, g_m\}$  of  $N$ , the following two statements are equivalent.

- (i)  $\mathcal{G}$  is a left Gröbner basis of  $N$  with respect to  $\prec_{e-gr}$ .
- (ii) Putting  $\sigma(\mathcal{G}) = \{\sigma(g_1), \dots, \sigma(g_m)\}$  and considering the filtration  $FN = \{F_q N = F_q L \cap N\}_{q \in \mathbb{N}}$  of  $N$  induced by  $FL$ ,  $\sigma(\mathcal{G})$  is a left Gröbner basis for the associated graded  $G(A)$ -module  $G(N)$  of  $N$  with respect to the graded left monomial ordering  $\prec_{\sigma(e)-gr}$  defined above.

**Proof** (i)  $\Rightarrow$  (ii) Note that any nonzero homogeneous element of  $G(N)$  is of the form  $\sigma(\xi)$  with  $\xi \in N$ . If  $\mathcal{G}$  is a left Gröbner basis of  $N$ , then there exists some  $g_i \in \mathcal{G}$  such that  $\mathbf{LM}(g_i) | \mathbf{LM}(\xi)$ , i.e., there is a monomial  $a^\alpha \in \mathcal{B}$  such that  $\mathbf{LM}(\xi) = \mathbf{LM}(a^\alpha \mathbf{LM}(g_i))$ . Since the given left monomial ordering  $\prec_{e-gr}$  on  $\mathcal{B}(e)$  is a graded left monomial ordering, it follows from Lemma 5.2.2 and Lemma 5.3.2 that

$$\begin{aligned}
 \mathbf{LM}(\sigma(\xi)) &= \sigma(\mathbf{LM}(\xi)) \\
 &= \sigma(\mathbf{LM}(a^\alpha \mathbf{LM}(g_i))) \\
 &= \mathbf{LM}(\sigma(a^\alpha \mathbf{LM}(g_i))) \\
 &= \mathbf{LM}(\sigma(a)^\alpha \sigma(\mathbf{LM}(g_i))) \\
 &= \mathbf{LM}(\sigma(a)^\alpha \mathbf{LM}(\sigma(g_i))).
 \end{aligned}$$

This shows that  $\mathbf{LM}(\sigma(g_i)) | \mathbf{LM}(\sigma(\xi))$ , thereby  $\sigma(\mathcal{G})$  is a left Gröbner basis for  $G(N)$ .

(ii)  $\Rightarrow$  (i) Suppose that  $\sigma(\mathcal{G})$  is a left Gröbner basis of  $G(N)$  with respect to  $\prec_{\sigma(e)-gr}$ . If  $\xi \in N$  and  $\xi \neq 0$ , then  $\sigma(\xi) \neq 0$ , and there exists a  $\sigma(g_i) \in \sigma(\mathcal{G})$  such that  $\mathbf{LM}(\sigma(g_i)) | \mathbf{LM}(\sigma(\xi))$ , i.e., there is a monomial  $\sigma(a)^\alpha \in \sigma(\mathcal{B})$  such that  $\mathbf{LM}(\sigma(\xi)) = \mathbf{LM}(\sigma(a)^\alpha \mathbf{LM}(\sigma(g_i)))$ . Again as  $\prec_{e-gr}$  is a left graded monomial ordering on  $\mathcal{B}(e)$ , by Lemma 5.2.2 and

Lemma 5.3.2 we have

$$\begin{aligned}
 \sigma(\mathbf{LM}(\xi)) &= \mathbf{LM}(\sigma(\xi)) \\
 &= \mathbf{LM}(\sigma(a)^\alpha \mathbf{LM}(\sigma(g_i))) \\
 &= \mathbf{LM}(\sigma(a)^\alpha \sigma(\mathbf{LM}(g_i))) \\
 &= \mathbf{LM}(\sigma(a^\alpha \mathbf{LM}(g_i))) \\
 &= \sigma(\mathbf{LM}(a^\alpha \mathbf{LM}(g_i))).
 \end{aligned}$$

This shows that  $d_{\text{fil}}(\mathbf{LM}(\xi)) = d_{\text{fil}}(\mathbf{LM}(a^\alpha \mathbf{LM}(g_i)))$ . Since both  $\mathbf{LM}(\xi)$  and  $\mathbf{LM}(a^\alpha \mathbf{LM}(g_i))$  are monomials in  $\mathcal{B}(e)$ , it follows from the construction of  $FL$  and Lemma 5.3.1 that  $\mathbf{LM}(\xi) = \mathbf{LM}(a^\alpha \mathbf{LM}(g_i))$ , i.e.,  $\mathbf{LM}(g_i) | \mathbf{LM}(\xi)$ . This shows that  $\mathcal{G}$  is a left Gröbner basis for  $N$ .  $\square$

Similarly, in light of Proposition 5.2.3 we may define an ordering  $\prec_{\tilde{e}}$  on the  $K$ -basis  $\mathcal{B}(e) = \{Z^m \tilde{a}^\alpha \tilde{e}_i \mid Z^m \tilde{a}^\alpha \in \tilde{\mathcal{B}}, 1 \leq i \leq s\}$  of the  $\mathbb{N}$ -graded free  $\tilde{A}$ -module  $\tilde{L} = \bigoplus_{i=1}^s \tilde{A} \tilde{e}_i$  subject to the rule: for  $Z^s \tilde{a}^\alpha \tilde{e}_i, Z^t \tilde{a}^\beta \tilde{e}_j \in \mathcal{B}(e)$ ,

$$Z^s \tilde{a}^\alpha \tilde{e}_i \prec_{\tilde{e}} Z^t \tilde{a}^\beta \tilde{e}_j \iff a^\alpha e_i \prec_{e-gr} a^\beta e_j, \text{ or } a^\alpha e_i = a^\beta e_j \text{ and } s < t,$$

where  $\prec_{e-gr}$  is a given graded left monomial ordering on  $\mathcal{B}(e)$ .

**5.3.4. Lemma** With the ordering  $\prec_{\tilde{e}}$  defined above, the following statements hold.

- (i)  $\prec_{\tilde{e}}$  is a left monomial ordering on  $\widetilde{\mathcal{B}(e)}$  (but not necessarily a graded left monomial ordering).
- (ii) (Compare with Corollary 5.1.6.)  $\mathbf{LM}(\tilde{\xi}) = \widetilde{\mathbf{LM}(\xi)}$  holds for all nonzero  $\xi \in L$ , where the monomial orderings used for taking  $\mathbf{LM}(\tilde{\xi})$  and  $\mathbf{LM}(\xi)$  are  $\prec_{\tilde{e}}$  and  $\prec_{e-gr}$  respectively.

**Proof** (i) Noticing that the given monomial ordering  $\prec_{gr}$  for  $A$  is a graded monomial ordering with respect to a positive-degree function  $d(\cdot)$  on  $A$ , it follows from Theorem 5.1.5(ii) that  $\tilde{A}$  is turned into an  $\mathbb{N}$ -graded solvable polynomial algebra by using the monomial ordering  $\prec_{\tilde{A}}$  defined on  $\tilde{\mathcal{B}}$  subject to the rule:

$$\tilde{a}^\alpha Z^s \prec_{\tilde{A}} \tilde{a}^\beta Z^t \iff a^\alpha \prec_{gr} a^\beta, \text{ or } a^\alpha = a^\beta \text{ and } s < t, \quad a^\alpha, a^\beta \in \mathcal{B},$$

where the positive-degree function on  $\tilde{A}$  is given by  $d(\tilde{a}_i) = d(a_i)$  for  $1 \leq i \leq n$ , and  $d(Z) = 1$ . Moreover, since  $\tilde{e}_i$  is a homogeneous element of degree  $b_i$  in  $\tilde{A}$ ,  $1 \leq i \leq s$ , by Lemma 5.2.2, it is then straightforward to verify that  $\prec_{\tilde{e}}$  is a left monomial ordering on  $\widetilde{\mathcal{B}(e)}$ .

(ii) Let  $\xi = \sum_{i,j} \lambda_{ij} a^{\alpha(i_j)} e_j$ , where  $\lambda_{ij} \in K^*$  and  $a^{\alpha(i_j)} \in \mathcal{B}$  with  $\alpha(i_j) = (\alpha_{i_{j1}}, \dots, \alpha_{i_{jn}}) \in \mathbb{N}^n$ . If  $d_{\text{fil}}(\xi) = q$ , i.e.,  $\xi \in F_q L - F_{q-1} L$ , then by Lemma 5.3.1,  $\mathbf{LM}(\xi) = a^{\alpha(i_t)} e_t$  for some  $t$  such that  $d_{\text{fil}}(a^{\alpha(i_t)} e_t) = d(a^{\alpha(i_t)}) + b_t = q$ . Since  $\prec_{e\text{-gr}}$  is a left graded monomial ordering on  $\mathcal{B}(e)$ , by Lemma 5.2.2 we have  $\xi = \lambda_{it} \widetilde{a}^{\alpha(i_t)} \widetilde{e}_t + \sum_{j \neq t} \lambda_{ij} Z^{q-\ell_{ij}} \widetilde{a}^{\alpha(i_j)} \widetilde{e}_j$ , where  $\ell_{ij} = d_{\text{fil}}(a^{\alpha(i_j)} e_j) = d(a^{\alpha(i_j)}) + d_j$ . It follows from the definition of  $\prec_{\widetilde{e}}$  that  $\mathbf{LM}(\widetilde{\xi}) = \widetilde{a}^{\alpha(i_t)} \widetilde{e}_t = \widetilde{\mathbf{LM}(\xi)}$ , as desired.  $\square$

**5.3.5. Theorem** Let  $N$  be a submodule of the filtered free  $A$ -module  $L = \bigoplus_{i=1}^s A e_i$ , where  $L$  is equipped with the filtration  $FL = \{F_q L\}_{q \in \mathbb{N}}$  such that  $d_{\text{fil}}(e_i) = b_i$ ,  $1 \leq i \leq s$ , and let  $\prec_{e\text{-gr}}$  be a graded left monomial ordering on  $\mathcal{B}(e)$ . For a subset  $\mathcal{G} = \{g_1, \dots, g_m\}$  of  $N$ , the following two statements are equivalent.

- (i)  $\mathcal{G}$  is a left Gröbner basis of  $N$  with respect to  $\prec_{e\text{-gr}}$ .
- (ii) Putting  $\tau(\mathcal{G}) = \{\widetilde{g}_1, \dots, \widetilde{g}_m\}$  and considering the filtration  $FN = \{F_q N = F_q L \cap N\}_{q \in \mathbb{N}}$  of  $N$  induced by  $FL$ ,  $\tau(\mathcal{G})$  is a left Gröbner basis for the Rees module  $\widetilde{N}$  of  $N$  with respect to the left monomial ordering  $\prec_{\widetilde{e}}$  defined before Lemma 5.3.4.

**Proof** (i)  $\Rightarrow$  (ii) Note that any nonzero homogeneous element of  $\widetilde{N}$  is of the form  $h_q(\xi)$  for some  $\xi \in F_q N$  with  $d_{\text{fil}}(\xi) = q_1 \leq q$ . By Lemma 5.2.2,  $h_q(\xi) = Z^{q-q_1} \widetilde{\xi}$ . If  $\mathcal{G}$  is a left Gröbner basis of  $N$ , then there exists some  $g_i \in \mathcal{G}$  such that  $\mathbf{LM}(g_i) | \mathbf{LM}(\xi)$ , i.e., there is a monomial  $a^\alpha \in \mathcal{B}$  such that  $\mathbf{LM}(\xi) = \mathbf{LM}(a^\alpha \mathbf{LM}(g_i))$ . It follows from Lemma 5.2.2 and Lemma 5.3.4 that

$$\begin{aligned} \mathbf{LM}(\widetilde{\xi}) &= \widetilde{\mathbf{LM}(\xi)} \\ &= (\mathbf{LM}(a^\alpha \mathbf{LM}(g_i)))^\sim \\ &= \mathbf{LM}((a^\alpha \mathbf{LM}(g_i))^\sim) \\ &= \mathbf{LM}(\widetilde{a}^\alpha \mathbf{LM}(g_i)) \\ &= \mathbf{LM}(\widetilde{a}^\alpha \mathbf{LM}(\widetilde{g}_i)). \end{aligned}$$

Hence, noticing the definition of  $\prec_{\widetilde{e}}$  we have

$$\begin{aligned} \mathbf{LM}(h_q(\xi)) &= \mathbf{LM}(Z^{q-q_1} \widetilde{\xi}) \\ &= Z^{q-q_1} \mathbf{LM}(\widetilde{\xi}) \\ &= Z^{q-q_1} \mathbf{LM}(\widetilde{a}^\alpha \mathbf{LM}(\widetilde{g}_i)) \\ &= \mathbf{LM}(z^{q-q_1} \widetilde{a}^\alpha \mathbf{LM}(\widetilde{g}_i)). \end{aligned}$$

This shows that  $\mathbf{LM}(\widetilde{g}_i) | \mathbf{LM}(h_q(\xi))$ , thereby  $\tau(\mathcal{G})$  is a left Gröbner basis of  $\widetilde{N}$ .

(ii)  $\Rightarrow$  (i) If  $\xi \in N$  and  $\xi \neq 0$ , then  $\tilde{\xi} \neq 0$  and  $\mathbf{LM}(\tilde{\xi}) = \widetilde{\mathbf{LM}(\xi)}$  by Lemma 5.3.4. Suppose that  $\tau(\mathcal{G})$  is a left Gröbner basis of  $\tilde{N}$  with respect to  $\prec_{\tilde{e}}$ . Then there exists some  $\tilde{g}_i \in \tau(\mathcal{G})$  such that  $\mathbf{LM}(\tilde{g}_i) | \mathbf{LM}(\tilde{\xi})$ , i.e., there is a monomial  $Z^m \tilde{a}^\gamma \in \tilde{\mathcal{B}}$  such that  $\mathbf{LM}(\tilde{\xi}) = \mathbf{LM}(Z^m \tilde{a}^\gamma \mathbf{LM}(\tilde{g}_i))$ . Since the given left monomial ordering  $\prec_{e-gr}$  on  $\mathcal{B}(e)$  is a graded left monomial ordering, it follows from Lemma 5.2.2, the definition of  $\prec_{\tilde{e}}$  and Lemma 5.3.2 that

$$\begin{aligned} \widetilde{a^\alpha e_j} = \widetilde{\mathbf{LM}(\xi)} = \mathbf{LM}(\tilde{\xi}) &= \mathbf{LM}(Z^m \tilde{a}^\gamma \mathbf{LM}(\tilde{g}_i)) \\ &= Z^m (\mathbf{LM}((a^\gamma \mathbf{LM}(g_i))^\sim)) \\ &= Z^m (\mathbf{LM}(a^\gamma \mathbf{LM}(g_i)))^\sim. \end{aligned}$$

Noticing the discussion on  $\tilde{L}$  and the role played by  $Z$  given before Lemma 5.2.2, we must have  $m = 0$ , thereby  $\mathbf{LM}(\xi) = \mathbf{LM}(a^\gamma \mathbf{LM}(g_i))$ . This shows that  $\mathcal{G}$  is a left Gröbner basis for  $N$ .

**Remark.** It is known that Gröbner bases for ungraded ideals in both a commutative polynomial algebra and a noncommutative free algebra can be obtained via computing homogeneous Gröbner bases for graded ideals in the corresponding homogenized (graded) algebras (cf. [Fröb], [Li2]). In our case here for an  $\mathbb{N}$ -filtered solvable polynomial algebra  $A$  with respect to a positive-degree function  $d(\cdot)$ , by using a (de)homogenization-like trick with respect to the central regular element  $Z$  in  $\tilde{A}$ , the discussion on  $\tilde{A}$  and  $\tilde{L}$  presented in previous Section 5.1 indeed enables us to have a whole theory and strategy similar to that given in (Section 3.6 and Section 3.7 of [Li2]), so that left Gröbner bases of submodules (left ideals) in  $L$  (in  $A$ ) can be obtained via computing homogeneous left Gröbner bases of graded submodules (graded left ideals) in  $\tilde{L}$  (in  $\tilde{A}$ ). Since such a topic is beyond the scope of this text, we omit the detailed discussion here.

## 5.4. F-Bases and Standard Bases with Respect to Good Filtration

Let  $A = K[a_1, \dots, a_n]$  be an  $\mathbb{N}$ -filtered solvable polynomial algebra with admissible system  $(\mathcal{B}, \prec)$  and the  $\mathbb{N}$ -filtration  $FA = \{F_p A\}_{p \in \mathbb{N}}$  constructed with respect to a given positive-degree function  $d(\cdot)$  on  $A$  (see Section 5.1). In this section, we introduce F-bases and standard bases respectively for  $\mathbb{N}$ -filtered left  $A$ -modules and their submodules with respect

to good filtration, and we show that any two minimal F-bases, respectively any two minimal standard bases have the same number of elements and the same number of elements of the same fil-degree. Moreover, we show that a standard basis for a submodule  $N$  of a filtered free  $A$ -module  $L$  can be obtained via computing a left Gröbner basis of  $N$  with respect to a graded left monomial ordering. All notions, notations and conventions used before are maintained.

Let  $M$  be a left  $A$ -module. We say that  $M$  is an  $\mathbb{N}$ -filtered  $A$ -module if  $M$  has a filtration  $FM = \{F_q M\}_{q \in \mathbb{N}}$ , where each  $F_q M$  is a  $K$ -subspace of  $M$ , such that  $M = \cup_{q \in \mathbb{N}} F_q M$ ,  $F_q M \subseteq F_{q+1} M$  for all  $q \in \mathbb{N}$ , and  $F_p A F_q M \subseteq F_{p+q} M$  for all  $p, q \in \mathbb{N}$ .

**Convention.** Unless otherwise stated, from now on in the subsequent sections a filtered  $A$ -module  $M$  is always meant an  $\mathbb{N}$ -filtered module with a filtration of the type  $FM = \{F_q M\}_{q \in \mathbb{N}}$  as described above.

Let  $G(A)$  be the associated graded algebra of  $A$ ,  $\tilde{A}$  the Rees algebra of  $A$ , and  $Z$  the homogeneous element of degree 1 in  $\tilde{A}_1$  represented by the multiplicative identity 1 of  $A$  (see Section 5.1). If  $M$  is a filtered  $A$ -module with the filtration  $FM = \{F_q M\}_{q \in \mathbb{N}}$ , then, actually as with a filtered free  $A$ -module in Section 5.2,  $M$  has the associated graded  $G(A)$ -module  $G(M) = \oplus_{q \in \mathbb{N}} G(M)_q$  with  $G(M)_0 = F_0 M$  and  $G(M)_q = F_q M / F_{q-1} M$  for  $q \geq 1$ , and the Rees module of  $M$  is defined as the graded  $\tilde{A}$ -module  $\tilde{M} = \oplus_{q \in \mathbb{N}} \tilde{M}_q$  with each  $\tilde{M}_q = F_q M$ . Also, we may define the *fil-degree* of a *nonzero*  $\xi \in M$ , that is,  $d_{\text{fil}}(\xi) = 0$  if  $\xi \in F_0 M$ , and if  $\xi \in F_q M - F_{q-1} M$  for some  $q > 0$ , then  $d_{\text{fil}}(\xi) = q$ . For a nonzero  $\xi \in M$  with  $d_{\text{fil}}(\xi) = q \geq 0$ , if we write  $\sigma(\xi)$  for the nonzero homogeneous element of degree  $q$  represented by  $\xi$  in  $G(M)_q$ ,  $\tilde{\xi}$  for the degree- $q$  homogeneous element represented by  $\xi$  in  $\tilde{M}_q$ , and  $h_{q'}(\xi)$  for the degree- $q'$  homogeneous element represented by  $\xi$  in  $\tilde{M}_{q'}$  with  $q < q'$ , then  $d_{\text{fil}}(\xi) = q = d_{\text{gr}}(\sigma(\xi)) = d_{\text{gr}}(\tilde{\xi})$ , and  $d_{\text{gr}}(h_{q'}(\xi)) = q'$ . Finally, as for a filtered free  $A$ -module in Section 5.2, we have  $\tilde{M}/Z\tilde{M} \cong G(M)$  as graded  $G(A)$ -modules, and  $\tilde{M}/(1 - Z)\tilde{M} \cong M$  as  $A$ -modules.

With notation as fixed above, the lemma presented below is a version of ([LVO], Ch.I, Lemma 5.4, Theorem 5.7) for  $\mathbb{N}$ -filtered modules.

**5.4.1. Lemma** Let  $M$  be a filtered  $A$ -module with the filtration  $FM =$

$\{F_q M\}_{q \in \mathbb{N}}$ , and  $V = \{v_1, \dots, v_m\}$  a finite subset of nonzero elements in  $M$ . The following statements are equivalent:

(i) There is a subset  $S = \{n_1, \dots, n_m\} \subset \mathbb{N}$  such that

$$F_q M = \sum_{i=1}^m \left( \sum_{p_i + n_i \leq q} F_{p_i} A \right) v_i, \quad q \in \mathbb{N};$$

(ii)  $G(M) = \sum_{i=1}^m G(A) \sigma(v_i)$ ;

(iii)  $\widetilde{M} = \sum_{i=1}^m \widetilde{A} \widetilde{v}_i$ .

□

**5.4.2. Definition** Let  $M$  be a filtered  $A$ -module with the filtration  $FM = \{F_q M\}_{q \in \mathbb{N}}$ , and let  $V = \{v_1, \dots, v_m\} \subset M$  be a finite subset of nonzero elements. If  $V$  satisfies one of the equivalent conditions of Lemma 5.4.1, then we call  $V$  an *F-basis* of  $M$  with respect to  $FM$ .

Let  $M$  be a filtered  $A$ -module with the filtration  $FM = \{F_q M\}_{q \in \mathbb{N}}$ . If  $V$  is an F-basis of  $M$  with respect to  $FM$ , then it is necessary to note that

- (1) since  $M = \cup_{q \in \mathbb{N}} F_q M$ , it is clear that  $V$  is certainly a generating set of the  $A$ -module  $M$ , i.e.,  $M = \sum_{i=1}^m A v_i$ ;
- (2) due to Lemma 5.4.1(i), the filtration  $FM$  is usually referred to as a *good filtration* of  $M$  in the literature concerning filtered module theory (cf. [LVO]).

Indeed, if  $M$  is a finitely generated  $A$ -module, then *any* finite generating set  $U = \{u_1, \dots, u_t\}$  of  $M$  can be turned into an F-basis with respect to some good filtration  $FM$ . More precisely, let  $S = \{n_1, \dots, n_t\}$  be an arbitrarily chosen subset of  $\mathbb{N}$ , then the required good filtration  $FM = \{F_q M\}_{q \in \mathbb{N}}$  can be defined by setting

$$\begin{aligned} F_q M &= \{0\} \text{ if } q < \min\{n_1, \dots, n_t\}; \\ F_q M &= \sum_{i=1}^t \left( \sum_{p_i + n_i \leq q} F_{p_i} A \right) u_i \text{ otherwise.} \end{aligned} \quad q \in \mathbb{N}.$$

In particular, if  $L = \oplus_{i=1}^s A e_i$  is a filtered free  $A$ -module with the filtration  $FL = \{F_q L\}_{q \in \mathbb{N}}$  as constructed in Subsection 5.2 such that  $d_{\text{fil}}(e_i) = b_i$ ,  $1 \leq i \leq s$ , then  $\{e_1, \dots, e_s\}$  is an F-basis of  $L$  with respect to the good filtration  $FL$ .

**5.4.3. Definition** Let  $M$  be a filtered  $A$ -module with the filtration  $FM = \{F_q M\}_{q \in \mathbb{N}}$ , and suppose that  $M$  has an F-basis  $V = \{v_1, \dots, v_m\}$  with respect to  $FM$ . If any proper subset of  $V$  cannot be an F-basis of  $M$  with respect to  $FM$ , then we say that  $V$  is a *minimal F-basis* of  $M$  with respect to  $FM$ .

Note that  $A$  is an  $\mathbb{N}$ -filtered  $K$ -algebra such that  $G(A) = \bigoplus_{p \in \mathbb{N}} G(A)_p$  with  $G(A)_0 = K$ ,  $\tilde{A} = \bigoplus_{p \in \mathbb{N}} \tilde{A}_p$  with  $\tilde{A}_0 = K$ , while  $K$  is a field. By Lemma 5.4.1 and the well-known result on graded modules over an  $\mathbb{N}$ -graded algebra with the degree-0 homogeneous part a field (cf. [Eis], Chapter 19; [Kr1], Chapter 3; [Li3]), we have immediately the following

**5.4.4. Proposition** Let  $M$  be a filtered  $A$ -module with the filtration  $FM = \{F_q M\}_{q \in \mathbb{N}}$ , and  $V = \{v_1, \dots, v_m\} \subset M$  a subset of nonzero elements. Then  $V$  is a minimal F-basis of  $M$  with respect to  $FM$  if and only if  $\sigma(V) = \{\sigma(v_1), \dots, \sigma(v_m)\}$  is a minimal homogeneous generating set of  $G(M)$  if and only if  $\tau(V) = \{\tilde{v}_1, \dots, \tilde{v}_m\}$  is a minimal homogeneous generating set of  $\tilde{M}$ . Hence, any two minimal F-bases of  $M$  with respect to  $FM$  have the same number of elements and the same number of elements of the same fil-degree. □

Let  $M$  be an  $\mathbb{N}$ -filtered  $A$ -module with the filtration  $FM = \{F_q M\}_{q \in \mathbb{N}}$ , and let  $N$  be a submodule of  $M$  with the filtration  $FN = \{F_q N = N \cap F_q M\}_{q \in \mathbb{N}}$  induced by  $FM$ . Then, as with a filtered free  $A$ -module in Section 5.2, the associated graded  $G(A)$ -module  $G(N) = \bigoplus_{q \in \mathbb{N}} G(N)_q$  of  $N$  with  $G(N)_q = F_q N / F_{q-1} N$  is a graded submodule of  $G(M)$ , and the Rees module  $\tilde{N} = \bigoplus_{q \in \mathbb{N}} \tilde{N}_q$  of  $N$  with  $\tilde{N}_q = F_q N$  is a graded submodule of  $\tilde{M}$ .

**5.4.5. Definition** Let  $M$  be a filtered  $A$ -module with the filtration  $FM = \{F_q M\}_{q \in \mathbb{N}}$ , and let  $N$  be a submodule of  $M$ . Consider the filtration  $FN = \{F_q N = N \cap F_q M\}_{q \in \mathbb{N}}$  of  $N$  induced by  $FM$ . If  $W = \{\xi_1, \dots, \xi_s\} \subset N$  is an F-basis with respect to  $FN$  in the sense of Definition 5.4.2, then we call  $W$  a *standard basis* of  $N$ .

**Remark.** By referring to Lemma 5.4.1, one may check that our definition 5.4.5 of a standard basis coincides with the classical Macaulay

basis provided  $A = K[x_1, \dots, x_n]$  is the commutative polynomial  $K$ -algebra (cf. [KR2], Definition 4.2.13, Theorem 4.3.19), for, taking the  $\mathbb{N}$ -filtration  $FA$  with respect to an arbitrarily chosen positive-degree function  $d(\cdot)$  on  $A$ , there are graded algebra isomorphisms  $G(A) \cong A$  and  $\tilde{A} \cong K[x_0, x_1, \dots, x_n]$ , where  $d(x_0) = 1$  and  $x_0$  plays the role that the central regular element  $Z$  of degree 1 does in  $\tilde{A}$ . Moreover, if two-sided ideals of an  $\mathbb{N}$ -filtered solvable polynomial algebra  $A$  are considered, then one may see that our definition 5.4.5 of a standard basis coincides with the standard basis defined in [Gol].

**5.4.6. Definition** Let  $M$  be a filtered  $A$ -module with the filtration  $FM = \{F_q M\}_{q \in \mathbb{N}}$ , and  $N$  a submodule of  $M$  with the filtration  $FN = \{F_q N = N \cap F_q M\}_{q \in \mathbb{N}}$  induced by  $FM$ . Suppose that  $N$  has a standard basis  $W = \{\xi_1, \dots, \xi_m\}$  with respect to  $FN$ . If any proper subset of  $W$  cannot be a standard basis for  $N$  with respect to  $FN$ , then we call  $W$  a *minimal standard basis* of  $N$  with respect to  $FN$ .

If  $N$  is a submodule of some filtered  $A$ -module  $M$  with filtration  $FM$ , then since a standard basis of  $N$  is defined as an  $F$ -basis of  $N$  with respect to the filtration  $FN$  induced by  $FM$ , the next proposition follows from Proposition 5.4.4.

**5.4.7. Proposition** Let  $M$  be a filtered  $A$ -module with the filtration  $FM = \{F_q M\}_{q \in \mathbb{N}}$ , and  $N$  a submodule of  $M$  with the induced filtration  $FN = \{F_q N = N \cap F_q M\}_{q \in \mathbb{N}}$ . A finite subset of nonzero elements  $W = \{\xi_1, \dots, \xi_s\} \subset N$  is a minimal standard basis of  $N$  with respect to  $FN$  if and only if  $\sigma(W) = \{\sigma(\xi_1), \dots, \sigma(\xi_m)\}$  is a minimal homogeneous generating set of  $G(N)$  if and only if  $\tau(W) = \{\tilde{\xi}_1, \dots, \tilde{\xi}_m\}$  is a minimal homogeneous generating set of  $\tilde{N}$ . Hence, any two minimal standard bases of  $N$  have the same number of elements and the same number of elements of the same fil-degree.

□

Since  $A$ ,  $G(A)$  and  $\tilde{A}$  are all Noetherian domains (Proposition 1.1.4(ii) of Chapter 1, Theorem 5.1.5 of the current chapter), if a filtered  $A$ -module  $M$  has an  $F$ -basis  $V$  with respect to a given filtration  $FM$ , then the existence of a standard basis for a submodule  $N$  of  $M$  follows immediately from Lemma 5.4.1. Our next theorem shows that a standard basis for a



submodule  $N$  of a filtered free  $A$ -module  $L$  can be obtained via computing a left Gröbner basis of  $N$  with respect to a *graded left monomial ordering*.

**5.4.8. Theorem** Let  $L = \bigoplus_{i=1}^s Ae_i$  be a filtered free  $A$ -module with the filtration  $FL = \{F_q L\}_{q \in \mathbb{N}}$  such that  $d_{\text{fil}}(e_i) = b_i$ ,  $1 \leq i \leq s$ , and let  $\prec_{e-gr}$  be a graded left monomial ordering on  $\mathcal{B}(e)$  (see Section 5.3). If  $\mathcal{G} = \{g_1, \dots, g_m\} \subset L$  is a left Gröbner basis for the submodule  $N = \sum_{i=1}^m Ag_i$  of  $L$  with respect to  $\prec_{e-gr}$ , then  $\mathcal{G}$  is a standard basis for  $N$  in the sense of Definition 5.4.5.

**Proof** If  $\xi \in F_q N = F_q L \cap N$  and  $\xi \neq 0$ , then  $d_{\text{fil}}(\xi) \leq q$  and  $\xi$  has a left Gröbner representation by  $\mathcal{G}$ , that is,  $\xi = \sum_{i,j} \lambda_{ij} a^{\alpha(i_j)} g_j$ , where  $\lambda_{ij} \in K^*$ ,  $a^{\alpha(i_j)} \in \mathcal{B}$  with  $\alpha(i_j) = (\alpha_{i_{j1}}, \dots, \alpha_{i_{jn}}) \in \mathbb{N}^n$ , satisfying  $\mathbf{LM}(a^{\alpha(i_j)} g_j) \preceq_{e-gr} \mathbf{LM}(\xi)$ . Suppose  $d_{\text{fil}}(g_j) = n_j$ ,  $1 \leq j \leq m$ . Since  $\prec_{e-gr}$  is a graded left monomial ordering on  $\mathcal{B}(e)$ , by Lemma 5.3.1 we may assume that  $\mathbf{LM}(g_j) = a^{\beta(j)} e_{t_j}$  with  $\beta(j) = (\beta_{j1}, \dots, \beta_{jn}) \in \mathbb{N}^n$  and  $1 \leq t_j \leq s$ , such that  $d(a^{\beta(j)}) + b_{t_j} = n_j$ , where  $d(\cdot)$  is the given positive-degree function on  $A$ . Furthermore, by (Lemma 2.1.2(ii) of Chapter 2), we have

$$\mathbf{LM}(a^{\alpha(i_j)} g_j) = \mathbf{LM}(a^{\alpha(i_j)} a^{\beta(j)} e_{t_j}) = a^{\alpha(i_j) + \beta(j)} e_{t_j},$$

and it follows from Lemma 5.1.2, Lemma 5.2.2 and Lemma 5.3.1 that  $d(a^{\alpha(i_j)}) + n_j = d(a^{\alpha(i_j)}) + d(a^{\beta(j)}) + b_{t_j} = d(a^{\alpha(i_j) + \beta(j)}) + b_{t_j} \leq q$ . Hence  $\xi \in \sum_{j=1}^m \left( \sum_{p_j + n_j \leq q} F_{p_j} A \right) g_j$ . This shows that  $F_q N = \sum_{j=1}^m \left( \sum_{p_j + n_j \leq q} F_{p_j} A \right) g_j$ , i.e.,  $\mathcal{G}$  is a standard basis for  $N$ .

## 5.5. Computation of Minimal F-Bases and Minimal Standard Bases

Let  $A = K[a_1, \dots, a_n]$  be an  $\mathbb{N}$ -filtered solvable polynomial algebra with admissible system  $(\mathcal{B}, \prec)$  and the  $\mathbb{N}$ -filtration  $FA = \{F_p A\}_{p \in \mathbb{N}}$  constructed with respect to a positive-degree function  $d(\cdot)$  on  $A$  (see Section 5.1). In this section we show how to algorithmically compute minimal F-bases for quotient modules of a filtered free left  $A$ -module  $L$ , and how to algorithmically compute minimal standard bases for submodules of  $L$  in the case where a graded left monomial ordering  $\prec_{e-gr}$  on  $L$  is employed. All notions, notations and conventions used before are maintained.

We start by a little more preparation. Let  $M$  and  $M'$  be  $\mathbb{N}$ -filtered left  $A$ -modules with the filtration  $FM = \{F_q M\}_{q \in \mathbb{N}}$  and  $FM' = \{F_q M'\}_{q \in \mathbb{N}}$  respectively, and  $M \xrightarrow{\varphi} M'$  an  $A$ -module homomorphism. If  $\varphi(F_q M) \subseteq F_q M'$  for all  $q \in \mathbb{N}$ , then we call  $\varphi$  a *filtered homomorphism*. In the literature, such filtered homomorphisms are also referred to as filtered homomorphism of degree-0 (cf. [NVO], [LVO]). By the definition it is clear that the identity map of  $\mathbb{N}$ -filtered  $A$ -modules is filtered homomorphism, and compositions of filtered homomorphisms are filtered homomorphisms. Thus, all  $\mathbb{N}$ -filtered left  $A$ -modules form a subcategory of the category of left  $A$ -modules, in which morphisms are the filtered homomorphisms as defined above. Furthermore, if  $M \xrightarrow{\varphi} M'$  is a filtered homomorphism with kernel  $\text{Ker} \varphi = N$ , then  $N$  is an  $\mathbb{N}$ -filtered submodule of  $M$  with the induced filtration  $FN = \{F_q N = N \cap F_q M\}_{q \in \mathbb{N}}$ , and the image  $\varphi(M)$  of  $\varphi$  is an  $\mathbb{N}$ -filtered submodule of  $M'$  with the filtration  $F\varphi(M) = \{F_q \varphi(M) = \varphi(F_q M)\}_{q \in \mathbb{N}}$ . Consequently, the exactness of a sequence  $N \xrightarrow{\varphi} M \xrightarrow{\psi} M'$  of filtered homomorphisms in the category of  $\mathbb{N}$ -filtered  $A$ -modules is defined as the same as for a sequence of usual  $A$ -module homomorphisms, i.e., the sequence satisfies  $\text{Im} \varphi = \text{Ker} \psi$ . Long exact sequence in the category of  $\mathbb{N}$ -filtered  $A$ -modules may be defined in an obvious way.

Let  $G(A)$  be the associated  $\mathbb{N}$ -graded algebra of  $A$  and  $\tilde{A}$  the Rees algebra of  $A$ . Then naturally, a filtered homomorphism  $M \xrightarrow{\varphi} M'$  induces a graded  $G(A)$ -module homomorphism  $G(M) \xrightarrow{G(\varphi)} G(M')$ , where if  $\xi \in F_q M$  and  $\bar{\xi} = \xi + F_{q-1} M$  is the coset represented by  $\xi$  in  $G(M)_q = F_q M / F_{q-1} M$ , then  $G(\varphi)(\bar{\xi}) = \varphi(\xi) + F_{q-1} M' \in G(M')_q = F_q M' / F_{q-1} M'$ , and  $\varphi$  induces a graded  $\tilde{A}$ -module homomorphism  $\tilde{M} \xrightarrow{\tilde{\varphi}} \tilde{M}'$ , where if  $\xi \in F_q M$  and  $h_q(\xi)$  is the homogeneous element of degree  $q$  in  $\tilde{M}_q = F_q M$ , then  $\tilde{\varphi}(h_q(\xi)) = h_q(\varphi(\xi)) \in \tilde{M}'_q = F_q M'$ . Moreover, if  $M \xrightarrow{\varphi} M' \xrightarrow{\psi} M''$  is a sequence of filtered homomorphisms, then  $G(\psi) \circ G(\varphi) = G(\psi \circ \varphi)$  and  $\tilde{\psi} \circ \tilde{\varphi} = \widetilde{\psi \circ \varphi}$ .

Furthermore, recall that a filtered homomorphism  $M \xrightarrow{\varphi} M'$  is called a *strict filtered homomorphism* if  $\varphi(F_q M) = \varphi(M) \cap F_q M'$  for all  $q \in \mathbb{N}$ . Note that if  $N$  is a submodule of  $M$  and  $\overline{M} = M/N$ , then, considering the induced filtration  $FN = \{F_q N = N \cap F_q M\}_{q \in \mathbb{N}}$  of  $N$  and the induced filtration  $F(\overline{M}) = \{F_q \overline{M} = (F_q M + N)/N\}_{q \in \mathbb{N}}$  of  $\overline{M}$ , the inclusion map  $N \hookrightarrow M$  and the canonical map  $M \rightarrow \overline{M}$  are strict filtered homomorphisms. Concerning strict filtered homomorphisms and

the induced graded homomorphisms, the next proposition is quoted from ([LVO], CH.I, Section 4).

**5.5.1. Proposition** Given a sequence of filtered homomorphisms

$$(*) \quad N \xrightarrow{\varphi} M \xrightarrow{\psi} M',$$

such that  $\psi \circ \varphi = 0$ , the following statements are equivalent.

- (i) The sequence  $(*)$  is exact and  $\varphi, \psi$  are strict filtered homomorphisms.
- (ii) The induced sequence  $G(N) \xrightarrow{G(\varphi)} G(M) \xrightarrow{G(\psi)} G(M')$  is exact.
- (iii) The induced sequence  $\widetilde{N} \xrightarrow{\widetilde{\varphi}} \widetilde{M} \xrightarrow{\widetilde{\psi}} \widetilde{M}'$  is exact.

□

Let  $L = \oplus_{i=1}^m A e_i$  be a filtered free  $A$ -module with the filtration  $FL = \{F_q L\}_{q \in \mathbb{N}}$  such that  $d_{\text{fil}}(e_i) = b_i$ ,  $1 \leq i \leq m$ . Then as we have noted in Section 5.4,  $\{e_1, \dots, e_m\}$  is an F-basis of  $L$  with respect to the good filtration  $FL$ . Let  $N$  be a submodule of  $L$ , and let the quotient module  $M = L/N$  be equipped with the filtration  $FM = \{F_q M = (F_q L + N)/N\}_{q \in \mathbb{N}}$  induced by  $FL$ . Without loss of generality, we assume that  $\bar{e}_i \neq 0$  for  $1 \leq i \leq m$ , where each  $\bar{e}_i$  is the coset represented by  $e_i$  in  $M$ . Then we see that  $\{\bar{e}_1, \dots, \bar{e}_m\}$  is an F-basis of  $M$  with respect to  $FM$ .

**5.5.2. Lemma** Let  $M = L/N$  be as fixed above, and let  $N = \sum_{j=1}^s A \xi_j$  be generated by the set of nonzero elements  $U = \{\xi_1, \dots, \xi_s\}$ , where  $\xi_\ell = \sum_{k=1}^s f_{k\ell} e_k$  with  $f_{k\ell} \in A$  and  $d_{\text{fil}}(\xi_\ell) = q_\ell$ ,  $1 \leq \ell \leq s$ . The following statements hold.

- (i) If for some  $j$ ,  $\xi_j$  has a nonzero term  $f_{ij} e_i$  such that  $d_{\text{fil}}(f_{ij} e_i) = d_{\text{fil}}(\xi_j) = q_j$  and the coefficient  $f_{ij}$  is a nonzero constant, say  $f_{ij} = 1$  without loss of generality, then for each  $\ell = 1, \dots, j-1, j+1, \dots, s$ , the element  $\xi'_\ell = \xi_\ell - f_{i\ell} \xi_j$  does not involve  $e_i$ . Putting  $U' = \{\xi'_1, \dots, \xi'_{j-1}, \xi'_{j+1}, \dots, \xi'_s\}$ ,  $N' = \sum_{\xi'_\ell \in U'} A \xi'_\ell$ , and considering the filtered free  $A$ -module  $L' = \oplus_{k \neq i} A e_k$  with the filtration  $FL' = \{F_q L'\}_{q \in \mathbb{N}}$  in which each  $e_k$  has the same filtered degree as it is in  $L$ , i.e.,  $d_{\text{fil}}(e_k) = b_k$ , if the quotient module  $M' = L'/N'$  is equipped with the filtration  $FM' = \{F_q M' = (F_q L' + N')/N'\}_{q \in \mathbb{N}}$  induced by  $FL'$ , then there is a strict filtered isomorphism  $\varphi: M' \cong M$ , i.e.,  $\varphi$  is an  $A$ -module isomorphism such that  $\varphi(F_q M') = F_q M$  for all  $q \in \mathbb{N}$ .

(ii) With the assumptions and notations as in (i), if  $U = \{\xi_1, \dots, \xi_s\}$  is a standard basis of  $N$  with respect to the filtration  $FN$  induced by  $FL$ , then  $U' = \{\xi'_1, \dots, \xi'_{j-1}, \xi'_{j+1}, \dots, \xi'_s\}$  is a standard basis of  $N'$  with respect to the filtration  $FN'$  induced by  $FL'$ .

**Proof** (i) Since  $f_{ij} = 1$  by the assumption, we see that every  $\xi'_\ell = \xi_\ell - f_{i\ell}\xi_j$  does not involve  $e_i$ . Let  $U' = \{\xi'_1, \dots, \xi'_{j-1}, \xi'_{j+1}, \dots, \xi'_s\}$  and  $N' = \sum_{\xi'_\ell \in U'} A\xi'_\ell$ . Then  $N' \subset L' = \oplus_{k \neq i} Ae_k \subset L$  and  $N = N' + A\xi_j$ . Since  $\xi_j = e_i + \sum_{k \neq i} f_{kj}e_k$ , the naturally defined  $A$ -module homomorphism  $M' = L'/N' \xrightarrow{\varphi} L/N = M$  with  $\varphi(\bar{e}_k) = \bar{e}_k$ ,  $k = 1, \dots, i-1, i+1, \dots, m$ , is an isomorphism, where, without confusion,  $\bar{e}_k$  denotes the coset represented by  $e_k$  in  $M'$  and  $M$  respectively. It remains to see that  $\varphi$  is a strict filtered isomorphism. Note that  $\{e_1, \dots, e_m\}$  is an F-basis of  $L$  with respect to  $FL$  such that  $d_{\text{fil}}(e_i) = b_i$ ,  $1 \leq i \leq m$ , i.e.,

$$F_q L = \sum_{i=1}^m \left( \sum_{p_i + b_i \leq q} F_{p_i} A \right) e_i, \quad q \in \mathbb{N},$$

that  $\{e_1, \dots, e_{i-1}, e_{i+1}, \dots, e_m\}$  is an F-basis of  $L'$  with respect to  $FL'$  such that  $d_{\text{fil}}(e_k) = b_k$ , where  $k \neq i$ , i.e.,

$$F_q L' = \sum_{k \neq i} \left( \sum_{p_i + b_k \leq q} F_{p_i} A \right) e_k, \quad q \in \mathbb{N},$$

and that  $\xi_j = e_i + \sum_{k \neq i} f_{kj}e_k$  with  $q_j = d_{\text{fil}}(\xi_j) = d_{\text{fil}}(e_i) = b_i$  such that  $d_{\text{fil}}(f_{kj}) + b_k \leq q_j$  for all  $f_{kj} \neq 0$ . It follows that  $\sum_{k \neq i} f_{kj}\bar{e}_k \in F_{q_j} M'$  and  $\varphi(\sum_{k \neq i} f_{kj}\bar{e}_k) = \bar{e}_i \in F_{q_j} M$ , thereby  $\varphi(F_q M') = F_q M$  for all  $q \in \mathbb{N}$ , as desired.

(ii) Note that  $\xi'_\ell = \xi_\ell - f_{i\ell}\xi_j$ . By the assumption on  $\xi_j$ , if  $f_{i\ell} \neq 0$  and  $d_{\text{fil}}(f_{i\ell}e_i) = d_{\text{fil}}(\xi_\ell) = q_\ell$ , then since  $d_{\text{fil}}(\xi_j) = d_{\text{fil}}(e_i)$  we have  $d_{\text{fil}}(f_{i\ell}\xi_j) = d_{\text{fil}}(\xi_\ell) = q_\ell$ . It follows that if we equip  $N$  with the filtration  $FN = \{F_q N = N \cap F_q L\}_{q \in \mathbb{N}}$  induced by  $FL$  and consider the associated graded module  $G(N)$  of  $N$ , then  $d_{\text{gr}}(\sigma(\xi_\ell)) = d_{\text{gr}}(\sigma(f_{i\ell}\xi_j)) = d_{\text{gr}}(\sigma(f_{i\ell})\sigma(\xi_j))$  in  $G(N)$ , i.e.,  $\sigma(\xi_\ell) - \sigma(f_{i\ell})\sigma(\xi_j) \in G(N)_{q_\ell}$ . So, if  $\sigma(\xi_\ell) - \sigma(f_{i\ell})\sigma(\xi_j) \neq 0$  then  $d_{\text{fil}}(\xi'_\ell) = q_\ell$  and thus

$$\sigma(\xi'_\ell) = \sigma(\xi_\ell - f_{i\ell}\xi_j) = \sigma(\xi_\ell) - \sigma(f_{i\ell})\sigma(\xi_j). \quad (1)$$

If  $f_{i\ell} \neq 0$  and  $d_{\text{fil}}(f_{i\ell}e_i) < d_{\text{fil}}(\xi_\ell) = q_\ell$ , then  $\sigma(\xi_\ell) = \sum_{d(f_{k\ell}) + b_k = q_\ell} \sigma(f_{k\ell})\sigma(e_k)$  does not involve  $\sigma(e_i)$ . Also since  $d_{\text{fil}}(\xi_j) = d_{\text{fil}}(e_i)$ , we have  $d_{\text{fil}}(f_{i\ell}\xi_j) <$

$d_{\text{fil}}(\xi_\ell) = q_\ell$ . Hence  $d_{\text{fil}}(\xi'_\ell) = d_{\text{fil}}(\xi_\ell) = q_\ell$  and thus

$$\sigma(\xi'_\ell) = \sigma(\xi_\ell - f_{i\ell}\xi_j) = \sigma(\xi_\ell). \quad (2)$$

If  $f_{i\ell} = 0$ , then the equality (1) is the same as equality (2). Now, if  $U = \{\xi_1, \dots, \xi_s\}$  is a standard basis of  $N$  with respect to the induced filtration  $FN$ , then  $G(N) = \sum_{\ell=1}^s G(A)\sigma(\xi_\ell)$  by Lemma 5.4.1. But since we have also  $G(N) = \sum_{\xi'_\ell \in U'} G(A)\sigma(\xi'_\ell) + G(A)\sigma(\xi_j)$  where the  $\sigma(\xi'_\ell)$  are those nonzero homogeneous elements obtained according to the above equalities (1) and (2), it follows from Lemma 5.4.1 that  $U' \cup \{\xi_j\}$  is a standard basis of  $N$  with respect to the induced filtration  $FN$ .

We next prove that  $U'$  is a standard basis of  $N' = \sum_{\xi'_\ell \in U'} A\xi'_\ell$  with respect to the filtration  $FN' = \{F_q N' = N' \cap F_q L'\}_{q \in \mathbb{N}}$  induced by  $FL'$ . Since  $\{e_1, \dots, e_{i-1}, e_{i+1}, \dots, e_m\}$  is an F-basis of  $L'$  with respect to the filtration  $FL'$  such that each  $e_k$  has the same filtered degree as it is in  $L$ , i.e.,  $d_{\text{fil}}(e_k) = b_k$ , it is clear that  $F_q L' = L' \cap F_q L$ ,  $q \in \mathbb{N}$ , i.e., the filtration  $FL'$  is the one induced by  $FL$ . Considering the filtration  $FN'$  of  $N'$  induced by  $FL'$ , it turns out that

$$F_q N' = N' \cap F_q L' = N' \cap F_q L \subseteq N \cap F_q L = F_q N, \quad q \in \mathbb{N}. \quad (3)$$

If  $\xi \in F_q N'$ , then since  $U' \cup \{\xi_j\}$  is a standard basis of  $N$  with respect to the induced filtration  $FN$ , the formula (3) entails that

$$\xi = \sum_{\xi'_\ell \in U'} f_\ell \xi'_\ell + f_j \xi_j \text{ with } f_\ell, f_j \in A, \quad d(f_\ell) + d_{\text{fil}}(\xi'_\ell) \leq q, \quad d(f_j) + d_{\text{fil}}(\xi_j) \leq q. \quad (4)$$

Note that every  $\xi'_\ell$  does not involve  $e_i$  and consequently  $\xi$  does not involve  $e_i$ . Hence  $f_j = 0$  in (4) by the assumption on  $\xi_j$ , and thus  $\xi \in \sum_{\xi'_\ell \in U'} \left( \sum_{p_i + q_\ell \leq q} F_{p_i} A \right) \xi'_\ell$ . Therefor we conclude that  $U'$  is a standard basis for  $N'$  with respect to the induced filtration  $FN'$ , as desired.  $\square$

Combining Proposition 5.4.4, we now show that for quotient modules of filtered free  $A$ -modules, a result similar to Proposition 4.3.11 holds true.

**5.5.3. Proposition** Let  $L = \bigoplus_{i=1}^m A e_i$ ,  $M = L/N$  be as in Lemma 5.5.2, and suppose that  $U = \{\xi_1, \dots, \xi_s\}$  is now a standard basis of  $N$  with

(iii)  $\{\bar{e}_{i_1}, \dots, \bar{e}_{i_{m'}}\}$  is a minimal F-basis of  $M$  with respect to the filtration  $FM$ .

### Algorithm-MINFB

BEGIN

END

END

**Proof** First note that for each  $\xi_\ell \in U$ ,  $d_{\text{fil}}(\xi_\ell)$  is determined by Lemma 5.2.1. Since the algorithm is clearly finite, the conclusions (i) and (ii) follow from Lemma 5.5.2.

To prove the conclusion (iii), by the strict filtered isomorphism  $M' = L'/N' \cong M$  (or the proof of Lemma 5.5.2(i)) it is sufficient to show that  $\{\bar{e}_{i_1}, \dots, \bar{e}_{i_{m'}}\}$  is a minimal F-basis of  $M'$  with respect to the filtration  $FM'$ . By the conclusion (ii),  $V = \{v_1, \dots, v_t\}$  is a standard basis of the submodule  $N' = \sum_{\ell=1}^t Av_\ell$  of  $L'$  with respect to the filtration  $FN'$  induced by  $FL'$  such that each  $v_\ell = \sum_{k=1}^{m'} h_{k\ell} e_{i_k}$  has the property that  $h_{k\ell} \in K^*$  implies  $d_{\text{fil}}(e_{i_k}) = b_{i_k} < d_{\text{fil}}(v_\ell)$ . It follows from Lemma 5.4.1 that  $G(N') = \sum_{k=1}^t G(A)\sigma(v_k)$  in which each  $\sigma(v_k) = \sum_{d(h_{k\ell})+b_{i_k}=d_{\text{fil}}(v_k)} \sigma(h_{k\ell})\sigma(e_{i_k})$  and all the coefficients  $\sigma(h_{k\ell})$  satisfy  $d_{\text{gr}}(\sigma(h_{k\ell})) > 0$  (please note the discussion in Section 5.2). Since  $G(A)_0 = K$ ,  $G(L') = \oplus_{k=1}^{m'} G(A)\sigma(e_{i_k})$  (Proposition 5.2.3) and  $G(N')$  is the graded syzygy module of the graded quotient module  $G(L')/G(N')$ , by the well-known result on finitely generated graded modules over  $\mathbb{N}$ -graded algebras with the degree-0 homogeneous part a field (cf. [Eis], Chapter 19; [Kr1], Chapter 3; [Li3]), we conclude that  $\{\overline{\sigma(e_{i_1})}, \dots, \overline{\sigma(e_{i_{m'}})}\}$  is a minimal homogeneous generating set of  $G(L')/G(N')$ . On the other hand, considering the naturally formed exact sequence of strict filtered homomorphisms

$$0 \longrightarrow N' \longrightarrow L' \longrightarrow M' = L'/N' \longrightarrow 0,$$

by Proposition 5.5.1 we have the  $\mathbb{N}$ -graded  $G(A)$ -module isomorphism  $G(L')/G(N') \cong G(L'/N') = G(M')$  with  $\overline{\sigma(e_{i_k})} \mapsto \sigma(\bar{e}_{i_k})$ ,  $1 \leq k \leq m'$ . Now, by means of Proposition 5.4.4, we conclude that  $\{\bar{e}_{i_1}, \dots, \bar{e}_{i_{m'}}\}$  is a minimal F-basis of  $M'$  with respect to the filtration  $FM'$ , as desired.  $\square$

Finally, let  $L = \oplus_{i=1}^s Ae_i$  be a filtered free  $A$ -module with the filtration  $FL = \{F_q L\}_{q \in \mathbb{N}}$  such that  $d_{\text{fil}}(e_i) = b_i$ ,  $1 \leq i \leq s$ , and let  $\prec_{e\text{-gr}}$  be a graded left monomial ordering on the  $K$ -basis  $\mathcal{B}(e) = \{a^\alpha e_i \mid a^\alpha \in \mathcal{B}, 1 \leq i \leq s\}$  of  $L$  (see Section 5.3). Combining Theorem 5.3.3 and (Theorem 4.3.8 of Chapter 4), the next theorem shows how to algorithmically compute a minimal standard basis.

**5.5.4. Theorem** Let  $N = \sum_{i=1}^c A\theta_i$  be a submodule of  $L$  generated by the subset of nonzero elements  $\Theta = \{\theta_1, \dots, \theta_c\}$ , and let  $FN = \{F_q N = F_q L \cap N\}_{q \in \mathbb{N}}$  be the filtration of  $N$  induced by  $FL$ . Then a minimal standard basis of  $N$  with respect to  $FN$  can be obtained by implementing the following procedures:

**Procudure 1.** With the initial input data  $\Theta = \{\theta_1, \dots, \theta_c\}$ , run

**Algorithm-LGB** presented in (Section 2.3 of Chapter 2) to compute a left Gröbner basis  $U = \{\xi_1, \dots, \xi_m\}$  for  $N$  with respect to  $\prec_{e-gr}$  on  $\mathcal{B}(e)$ .

**Procudure 2.** Let  $G(N)$  be the associated graded  $G(A)$ -module of  $N$  determined by the induced filtration  $FN$ . Then  $G(N)$  is a graded submodule of the associated grade free  $G(A)$ -module  $G(L)$ , and it follows from Theorem 5.3.3 that  $\sigma(U) = \{\sigma(\xi_1), \dots, \sigma(\xi_m)\}$  is a homogeneous left Gröbner basis of  $G(N)$  with respect to  $\prec_{\sigma(e)-gr}$  on  $\sigma(\mathcal{B}(e))$ . With the initial input data  $\sigma(U)$ , run **Algorithm-MINHGS** presented in (Theorem 4.3.8 of Chapter 4) to compute a minimal homogeneous generating set  $\{\sigma(\xi_{j_1}), \dots, \sigma(\xi_{j_t})\} \subseteq \sigma(U)$  for  $G(N)$ .

**Procudure 3.** Write down  $W = \{\xi_{j_1}, \dots, \xi_{j_t}\}$ . Then  $W$  is a Minimal standard basis for  $N$  by Proposition 5.4.7.

□

**Remark.** (i) Note that the initial input data  $\sigma(U)$  in **Procedure 2** above is already a left Gröbner basis for  $G(N)$ . By (Theorem 4.3.8 of Chapter 4), the finally returned left Gröbner basis by **Algorithm-MINHGS** is indeed a minimal left Gröbner basis for  $G(N)$ .

(ii) By Theorem 5.3.5 and Proposition 5.4.7 it is clear that we can also obtain a minimal standard basis of the submodule  $N$  via computing a minimal homogeneous generating set for the Rees module  $\tilde{N}$  of  $N$ , which is a graded submodule of the Rees module  $\tilde{L}$  of  $L$ . However, noticing the structure of  $\tilde{A}$  and  $\tilde{L}$  (see Theorem 5.1.5, Proposition 5.2.3) it is equally clear that the cost of working on  $\tilde{A}$  will be much higher than working on  $G(N)$ .

## 5.6. Minimal Filtered Free Resolutions and Their Uniqueness

Let  $A = K[a_1, \dots, a_n]$  be an  $\mathbb{N}$ -filtered solvable polynomial algebra with admissible system  $(\mathcal{B}, \prec)$  and the  $\mathbb{N}$ -filtration  $FA = \{F_p A\}_{p \in \mathbb{N}}$  constructed with respect to a positive-degree function  $d(\cdot)$  on  $A$  (see Section 5.1). In this section, by using minimal F-bases and minimal standard bases in the sense of Definition 5.4.3 and Definition 5.4.6, we define minimal filtered free resolutions for finitely generated left  $A$ -modules, and we show that such minimal resolutions are unique up to strict filtered isomorphism of chain complexes in the category of filtered  $A$ -modules. All



notions, notations and conventions used before are maintained.

Let  $M = \sum_{i=1}^m A\xi_i$  be an arbitrary finitely generated  $A$ -module. Then, as we have noted in section 5.4,  $M$  may be endowed with a good filtration  $FM = \{F_q M\}_{q \in \mathbb{N}}$  with respect to an arbitrarily chosen subset  $\{n_1, \dots, n_m\} \subset \mathbb{N}$ , where

$$\begin{aligned} F_q M &= \{0\} \text{ if } q < \min\{n_1, \dots, n_m\}; \\ F_q M &= \sum_{i=1}^t \left( \sum_{p_i + n_i \leq q} F_{p_i} A \right) \xi_i \text{ otherwise,} \end{aligned} \quad q \in \mathbb{N}.$$

**5.6.1. Proposition** Let  $M$  and  $FM$  be as above. Consider the filtered free  $A$ -module  $L = \oplus_{i=1}^m Ae_i$  with the good filtration  $FL = \{F_q L\}_{q \in \mathbb{N}}$  such that  $d_{\text{fil}}(e_i) = n_i$ ,  $1 \leq i \leq m$ , and the exact sequence of  $A$ -module homomorphisms

$$(*) \quad 0 \longrightarrow N \xrightarrow{\iota} L \xrightarrow{\varphi} M \longrightarrow 0,$$

in which  $\varphi(e_i) = \xi_i$ ,  $1 \leq i \leq m$ ,  $N = \text{Ker} \varphi$  with the induced filtration  $FN = \{F_q N = N \cap F_q L\}_{q \in \mathbb{N}}$ , and  $\iota$  is the inclusion map. The following statements hold.

(i) The  $A$ -module homomorphisms  $\iota$  and  $\varphi$  are strict filtered homomorphisms. Equipping  $\overline{L} = L/N$  with the induced filtration  $F\overline{L} = \{F_q \overline{L} = (F_q L + N)/N\}_{q \in \mathbb{N}}$ , the induced  $A$ -module isomorphism  $\overline{L} \xrightarrow{\overline{\varphi}} M$  is a strict filtered isomorphism, that is,  $\overline{L} \cong M$  and  $\overline{\varphi}$  satisfies  $\overline{\varphi}(F_q \overline{L}) = F_q M$  for all  $q \in \mathbb{N}$ .

(ii) The induced sequence

$$G(*) \quad 0 \longrightarrow G(N) \xrightarrow{G(\iota)} G(L) \xrightarrow{G(\varphi)} G(M) \longrightarrow 0$$

is an exact sequence of graded  $G(A)$ -module homomorphisms, thereby  $G(L)/G(N) \cong G(M) \cong G(\overline{L}) = G(L/N)$  as graded  $G(A)$ -modules.

(iii) The induced sequence

$$(*) \quad 0 \longrightarrow \widetilde{N} \xrightarrow{\widetilde{\iota}} \widetilde{L} \xrightarrow{\widetilde{\varphi}} \widetilde{M} \longrightarrow 0$$

is an exact sequence of graded  $\widetilde{A}$ -module homomorphisms, thereby  $\widetilde{L}/\widetilde{N} \cong \widetilde{M} \cong \widetilde{\overline{L}} = \widetilde{L/N}$  as graded  $\widetilde{A}$ -modules.

**Proof** By the construction of  $FL$  (see section 5.2) and Proposition 5.5.1, the proof of all assertions is a straightforward exercise.  $\square$

Proposition 5.6.1(i) enables us to make the following

**Convention.** In what follows we shall always assume that a finitely generated  $A$ -module  $M$  is of the form as presented in Proposition 5.6.1(i), i.e.,  $M = L/N$ , and  $M$  has the good filtration

$$FM = \{F_q M = (F_q L + N)/N\}_{q \in \mathbb{N}}.$$

Comparing with the classical minimal graded free resolutions defined for finitely generated graded modules over finitely generated  $\mathbb{N}$ -graded algebras with the degree-0 homogeneous part a field (cf. [Eis], Chapter 19; [Kr1], Chapter 3; [Li3]), the results obtained in previous sections and the preliminary we made above naturally motivate the following

**5.6.2. Definition** Let  $L_0 = \bigoplus_{i=1}^m A e_i$  be a filtered free  $A$ -module with the filtration  $FL_0 = \{F_q L_0\}_{q \in \mathbb{N}}$  such that  $d_{\text{fil}}(e_i) = b_i$ ,  $1 \leq i \leq m$ , let  $N$  be a submodule of  $L_0$ , and let the  $A$ -module  $M = L_0/N$  be equipped with the filtration  $FM = \{F_q M = (F_q L_0 + N)/N\}_{q \in \mathbb{N}}$ . A *minimal filtered free resolution* of  $M$  is an exact sequence of filtered  $A$ -modules and filtered homomorphisms

$$\mathcal{L}_\bullet \quad \cdots \xrightarrow{\varphi_{i+1}} L_i \xrightarrow{\varphi_i} \cdots \xrightarrow{\varphi_2} L_1 \xrightarrow{\varphi_1} L_0 \xrightarrow{\varphi_0} M \rightarrow 0$$

satisfying

- (1)  $\varphi_0$  is the canonical epimorphism, i.e.,  $\varphi_0(e_i) = \bar{e}_i$  for  $e_i \in \mathcal{E}_0 = \{e_1, \dots, e_m\}$  (where each  $\bar{e}_i$  denotes the coset represented by  $e_i$  in  $M$ ), such that  $\varphi_0(\mathcal{E}_0)$  is a minimal F-basis of  $M$  with respect to  $FM$  (in the sense of Definition 5.4.3);
- (2) for  $i \geq 1$ , each  $L_i$  is a filtered free  $A$ -module with finite  $A$ -basis  $\mathcal{E}_i$ , and each  $\varphi_i$  is a strict filtered homomorphism, such that  $\varphi_i(\mathcal{E}_i)$  is a minimal standard basis of  $\text{Ker} \varphi_{i-1}$  with respect to the filtration induced by  $FL_{i-1}$  (in the sense of Definition 5.4.6).

To see that the minimal filtered free resolution introduced above is an appropriate definition of “minimal free resolution” for finitely generated modules over the  $\mathbb{N}$ -filtered solvable polynomial algebras with filtration determined by positive-degree functions, we now show that such a resolution is unique up to a strict filtered isomorphism of chain complexes in the category of  $\mathbb{N}$ -filtered  $A$ -modules.

**5.6.3. Theorem** Let  $\mathcal{L}_\bullet$  be a minimal filtered free resolution of  $M$  as presented in Definition 5.6.2. The following statements hold.

(i) The induced sequence of graded  $G(A)$ -modules and graded  $G(A)$ -module homomorphisms

$$G(\mathcal{L}_\bullet) \quad \cdots \xrightarrow{G(\varphi_{i+1})} G(L_i) \xrightarrow{G(\varphi_i)} \cdots \xrightarrow{G(\varphi_2)} G(L_1) \xrightarrow{G(\varphi_1)} G(L_0) \xrightarrow{G(\varphi_0)} G(M) \rightarrow 0$$

is a minimal graded free resolution of the finitely generated graded  $G(A)$ -module  $G(M)$ .

(ii) The induced sequence of graded  $\tilde{A}$ -modules and graded  $\tilde{A}$ -module homomorphisms

$$\tilde{\mathcal{L}}_\bullet \quad \cdots \xrightarrow{\tilde{\varphi}_{i+1}} \tilde{L}_i \xrightarrow{\tilde{\varphi}_i} \cdots \xrightarrow{\tilde{\varphi}_2} \tilde{L}_1 \xrightarrow{\tilde{\varphi}_1} \tilde{L}_0 \xrightarrow{\tilde{\varphi}_0} \tilde{M} \rightarrow 0$$

is a minimal graded free resolution of the finitely generated graded  $\tilde{A}$ -module  $\tilde{M}$ .

(iii)  $\mathcal{L}_\bullet$  is uniquely determined by  $M$  in the sense that if  $M$  has another minimal filtered free resolution

$$\mathcal{L}'_\bullet \quad \cdots \xrightarrow{\varphi'_{i+1}} L'_i \xrightarrow{\varphi'_i} \cdots \xrightarrow{\varphi'_2} L'_1 \xrightarrow{\varphi'_1} L_0 \xrightarrow{\varphi_0} M \rightarrow 0$$

then for each  $i \geq 1$ , there is a strict filtered  $A$ -module isomorphism  $\chi_i: L_i \rightarrow L'_i$  such that the diagram

$$\begin{array}{ccc} L_i & \xrightarrow{\varphi_i} & L_{i-1} \\ \chi_i \downarrow \cong & & \chi_{i-1} \downarrow \cong \\ L'_i & \xrightarrow{\varphi'_i} & L'_{i-1} \end{array}$$

is commutative, thereby  $\{\chi_i \mid i \geq 1\}$  gives rise to a strict filtered isomorphism of chain complexes  $\mathcal{L}_\bullet \cong \mathcal{L}'_\bullet$  in the category of  $\mathbb{N}$ -filtered  $A$ -modules.

**Proof** (i) and (ii) follow from Proposition 5.4.4, Proposition 5.4.7, and Proposition 5.6.1.

To prove (iii), let the sequence  $\mathcal{L}'_\bullet$  be as presented above. By the assertion (i),  $G(\mathcal{L}'_\bullet)$  is another minimal graded free resolution of  $G(M)$ . It follows from the well-known result on minimal graded free resolutions ([Eis], Chapter 19; [Kr1], Chapter 3; [Li3]) that there is a graded isomorphism of chain complexes  $G(\mathcal{L}_\bullet) \cong G(\mathcal{L}'_\bullet)$  in the category of graded

$G(A)$ -modules, i.e., for each  $i \geq 1$ , there is a graded  $G(A)$ -modules isomorphism  $\psi_i: G(L_i) \rightarrow G(L'_i)$  such that the diagram

$$\begin{array}{ccc} G(L_i) & \xrightarrow{G(\varphi_i)} & G(L_{i-1}) \\ \psi_i \downarrow \cong & & \psi_{i-1} \downarrow \cong \\ G(L'_i) & \xrightarrow{G(\varphi'_i)} & G(L'_{i-1}) \end{array}$$

is commutative. Our aim below is to construct the desired strict filtered isomorphisms  $\chi_i$  by using the graded isomorphisms  $\psi_i$  carefully. So, starting with  $L_0$ , we assume that we have constructed the strict filtered isomorphisms  $L_j \xrightarrow{\chi_j} L'_j$ , such that  $G(\chi_j) = \psi_j$  and  $\chi_{j-1}\varphi_j = \varphi'_j\chi_j$ ,  $1 \leq j \leq i-1$ . Let  $L_i = \bigoplus_{j=1}^{s_i} Ae_{ij}$ . Since each  $\psi_i$  is a graded isomorphism, we have  $\psi_i(\sigma(e_{ij})) = \sigma(\xi'_j)$  for some  $\xi'_j \in L'_i$  satisfying  $d_{\text{fil}}(\xi'_j) = d_{\text{gr}}(\sigma(\xi'_j)) = d_{\text{gr}}(\sigma(e_{ij})) = d_{\text{fil}}(e_{ij})$ . It follows that if we construct the filtered  $A$ -module homomorphism  $L_i \xrightarrow{\tau_i} L'_i$  by setting  $\tau_i(e_{ij}) = \xi'_j$ ,  $1 \leq j \leq s_i$ , then  $G(\tau_i) = \psi_i$ . Hence,  $\tau_i$  is a strict filtered isomorphism by Proposition 5.5.1. Since  $\psi_{i-1} = G(\chi_{i-1})$ ,  $\psi_i = G(\tau_i)$ , and thus

$$\begin{aligned} \psi_{i-1}G(\varphi_i) &= G(\chi_{i-1})G(\varphi_i) = G(\chi_{i-1}\varphi_i) \\ G(\varphi'_i)\psi_i &= G(\varphi'_i)G(\tau_i) = G(\varphi'_i\tau_i), \end{aligned}$$

for each  $q \in \mathbb{N}$ , by the strictness of the  $\varphi_j$ ,  $\varphi'_j$ ,  $\chi_j$  and  $\tau_i$ , we have

$$\begin{aligned} G(\chi_{i-1}\varphi_i)(G(L_i)_q) &= ((\chi_{i-1}\varphi_i)(F_q L_i) + F_{q-1}L'_{i-1})/F_{q-1}L'_{i-1} \\ &= (\chi_{i-1}(\varphi_i(L_i) \cap F_q L_{i-1}) + F_{q-1}L'_{i-1})/F_{q-1}L'_{i-1} \\ &\subseteq ((\chi_{i-1}\varphi_i)(L_{i-1}) \cap \chi_{i-1}(F_q L_{i-1}) + F_{q-1}L'_{i-1})/F_{q-1}L'_{i-1} \\ &= ((\chi_{i-1}\varphi_i)(L_i) \cap F_q L'_{i-1} + F_{q-1}L'_{i-1})/F_{q-1}L'_{i-1}, \end{aligned}$$

$$\begin{aligned} G(\varphi'_i\tau_i)(G(L_i)_q) &= ((\varphi'_i\tau_i)(F_q L_i) + F_{q-1}L'_{i-1})/F_{q-1}L'_{i-1} \\ &= (\varphi'_i(F_q L'_i) + F_{q-1}L'_{i-1})/F_{q-1}L'_{i-1} \\ &= (\varphi'_i(L'_i) \cap F_q L'_{i-1} + F_{q-1}L'_{i-1})/F_{q-1}L'_{i-1} \\ &= ((\varphi'_i\tau_i)(L_i) \cap F_q L'_{i-1} + F_{q-1}L'_{i-1})/F_{q-1}L'_{i-1}. \end{aligned}$$

Note that by the exactness of  $\mathcal{L}_\bullet$  and  $\mathcal{L}'_\bullet$ , as well as the commutativity  $\chi_{i-2}\varphi_{i-1} = \varphi'_{i-1}\chi_{i-1}$ , we have  $(\chi_{i-1}\varphi_i)(L_i) \subseteq \varphi'_i(L'_i) = (\varphi'_i\tau_i)(L_i)$ . Considering the filtration of the submodules  $(\chi_{i-1}\varphi_i)(L_i)$  and  $(\varphi'_i\tau_i)(L_i)$  induced by the filtration  $FL'_{i-1}$  of  $L'_{i-1}$ , the commutativity  $\psi_{i-1}G(\varphi_i) = G(\varphi'_i)\psi_i$  and the formulas derived above show

that both submodules have the same associated graded module, i.e.,  $G((\chi_{i-1}\varphi_i)(L_i)) = G((\varphi'_i\tau_i)(L_i))$ . It follows from a similar proof of ([LVO], Theorem 5.7 on P.49) that

$$(\chi_{i-1}\varphi_i)(L_i) = (\varphi'_i\tau_i)(L_i). \quad (1)$$

Clearly, the equality (1) does not necessarily mean the commutativity of the diagram

$$\begin{array}{ccc} L_i & \xrightarrow{\varphi_i} & L_{i-1} \\ \tau_i \downarrow \cong & & \chi_{i-1} \downarrow \cong \\ L'_i & \xrightarrow{\varphi'_i} & L'_{i-1} \end{array}$$

To remedy this problem, we need to further modify the filtered isomorphism  $\tau_i$ . Since  $G(\chi_{i-1}\varphi_i)(\sigma(e_{i_j})) = G(\varphi'_i\tau_i)(\sigma(e_{i_j}))$ ,  $1 \leq j \leq s_i$ , if  $d_{\text{fil}}(e_{i_j}) = b_j$ , then by the equality (1) and the strictness of  $\tau_i$  and  $\varphi_i$  we have

$$\begin{aligned} (\chi_{i-1}\varphi_i)(e_{i_j}) - (\varphi'_i\tau_i)(e_{i_j}) &\in (\varphi'_i\tau_i)(L_i) \cap F_{b_j-1}L'_{i-1} \\ &= \varphi'_i(L'_i) \cap F_{b_j-1}L'_{i-1} \\ &= \varphi'_i(F_{b_j-1}L'_i) \\ &= (\varphi'_i\tau_i)(F_{b_j-1}L_i), \end{aligned} \quad (2)$$

and furthermore from (2) we have a  $\xi_j \in F_{b_j-1}L_i$  such that  $d_{\text{fil}}(e_{i_j} - \xi_j) = b_j$  and

$$(\chi_{i-1}\varphi_i)(e_{i_j}) = (\varphi'_i\tau_i)(e_{i_j} - \xi_j), \quad 1 \leq j \leq s_i. \quad (3)$$

Now, if we construct the filtered homomorphism  $L_i \xrightarrow{\chi_i} L'_i$  by setting  $\chi_i(e_{i_j}) = \tau_i(e_{i_j} - \xi_j)$ ,  $1 \leq j \leq s_i$ , then since  $\tau_i(\xi_j) \in F_{b_j-1}L'_i$ , it turns out that

$$G(\chi_i)(\sigma(e_{i_j})) = G(\tau_i)(\sigma(e_{i_j} - \xi_j)) = G(\tau_i)(\sigma(e_{i_j})) = \psi_i(\sigma(e_{i_j})), \quad 1 \leq j \leq s_i,$$

thereby  $G(\chi_i) = \psi_i$ . Hence,  $\chi_i$  is a strict filtered isomorphism by Proposition 5.5.1, and moreover, it follows from (3) that we have reached the following diagram

$$\begin{array}{ccccccc} \cdots & \xrightarrow{\varphi_{i+1}} & L_i & \xrightarrow{\varphi_i} & L_{i-1} & \xrightarrow{\varphi_{i-1}} & \cdots \\ & & \chi_i \downarrow \cong & & \chi_{i-1} \downarrow \cong & & \\ \cdots & \xrightarrow{\varphi'_{i+1}} & L'_i & \xrightarrow{\varphi'_i} & L'_{i-1} & \xrightarrow{\varphi'_{i-1}} & \cdots \end{array}$$

in which  $\chi_{i-1}\varphi_i = \varphi'_i\chi_i$ . Repeating the same process to getting the desired  $\chi_{i+1}$  and so on, the proof is thus finished.

## 5.7. Computation of Minimal Finite Filtered Free Resolutions

Let  $A = K[a_1, \dots, a_n]$  be a solvable polynomial algebra with the admissible system  $(\mathcal{B}, \prec_{gr})$  in which  $\prec_{gr}$  is a graded monomial ordering with respect to some given positive-degree function  $d(\cdot)$  on  $A$  (see section 1.1 of Chapter 1). Thereby  $A$  is turned into an  $\mathbb{N}$ -filtered solvable polynomial algebra with the filtration  $FA = \{F_p A\}_{p \in \mathbb{N}}$  constructed with respect to the same  $d(\cdot)$  (see Example (2) given in Section 5.1). Note that Theorem 3.1.1, Theorem 3.1.2, and Theorem 3.2.2 given in Chapter 3 hold true for *any* solvable polynomial algebra. Combining the results of Chapter 4 and previous sections of the current chapter, we are now able to work out the algorithmic procedures for computing minimal finite filtered free resolutions over  $A$  (in the sense of Definition 5.6.2) with respect to any graded left monomial ordering on free left modules. All notions, notations and conventions used before are maintained.

**5.7.1. Theorem** Let  $L_0 = \bigoplus_{i=1}^m A e_i$  be a filtered free  $A$ -module with the filtration  $FL_0 = \{F_q L_0\}_{q \in \mathbb{N}}$  such that  $d_{\text{fil}}(e_i) = b_i$ ,  $1 \leq i \leq m$ . If  $N = \sum_{i=1}^s A \xi_i$  is a finitely generated submodule of  $L_0$  and the quotient module  $M = L_0/N$  is equipped with the filtration  $FM = \{F_q M = (F_q L_0 + N)/N\}_{q \in \mathbb{N}}$ , then  $M$  has a minimal filtered free resolution of length  $d \leq n$ :

$$\mathcal{L}_\bullet \quad 0 \longrightarrow L_d \xrightarrow{\varphi_d} \dots \xrightarrow{\varphi_2} L_1 \xrightarrow{\varphi_1} L_0 \xrightarrow{\varphi_0} M \longrightarrow 0$$

which can be constructed by implementing the following procedures:

**Procedure 1.** Fix a graded left monomial ordering  $\prec_{e-gr}$  on the  $K$ -basis  $\mathcal{B}(e)$  of  $L_0$  (see Section 5.3), and run **Algorithm-LGB** (presented in Section 2.3 of Chapter 2) with the initial input data  $U = \{\xi_1, \dots, \xi_s\}$  to compute a left Gröbner basis  $\mathcal{G} = \{g_1, \dots, g_z\}$  for  $N$ , so that  $N$  has the standard basis  $\mathcal{G}$  with respect to the induced filtration  $FN = \{F_q N = N \cap F_q L_0\}_{q \in \mathbb{N}}$  (Theorem 5.4.8).

**Procedure 2.** Run **Algorithm-MINFB** (presented in Proposition 5.5.3) with the initial input data  $E = \{e_1, \dots, e_m\}$  and  $\mathcal{G} = \{g_1, \dots, g_z\}$  to compute a subset  $\mathcal{E}'_0 = \{e_{i_1}, \dots, e_{i_{m'}}\} \subset \mathcal{E}_0 = \{e_1, \dots, e_m\}$  and a subset  $V = \{v_1, \dots, v_t\} \subset N \cap L'_0$  such that there is a strict filtered isomorphism  $L'_0/N' = M' \cong M$ , where  $L'_0 = \bigoplus_{q=1}^{m'} A e_{i_q}$  with  $m' \leq m$  and  $N' = \sum_{k=1}^t A v_k$ , and such that  $\{\bar{e}_{i_1}, \dots, \bar{e}_{i_{m'}}\}$  is a minimal F-basis of  $M$  with respect to the filtration  $FM$ .

For convenience, after accomplishing Procedure 2 we may assume that  $\mathcal{E}_0 = \mathcal{E}_0'$ ,  $U = V$  and  $N = N'$ . Accordingly we have the short exact sequence

$$0 \longrightarrow N \xrightarrow{\iota} L_0 \xrightarrow{\varphi_0} M \longrightarrow 0$$

such that  $\varphi_0(\mathcal{E}_0) = \{\bar{e}_1, \dots, \bar{e}_m\}$  is a minimal F-basis of  $M$  with respect to the filtration  $FM$ , where  $\iota$  is the inclusion map.

**Procedure 3.** With the initial input data  $U = V$ , implements the procedures presented in Theorem 5.5.4 to compute a minimal standard basis  $W = \{\xi_{j_1}, \dots, \xi_{j_{m_1}}\}$  for  $N$  with respect to the induced filtration  $FN$ .

**Procedure 4.** Computes a generating set  $U_1 = \{\eta_1, \dots, \eta_{s_1}\}$  of  $N_1 = \text{Syz}(W)$  in the free  $A$ -module  $L_1 = \bigoplus_{i=1}^{m_1} A\varepsilon_i$  by running **Algorithm-LGB** with the initial input data  $W$  and using Theorem 3.1.2.

**Procedure 5.** Construct the strict filtered exact sequence

$$0 \longrightarrow N_1 \longrightarrow L_1 \xrightarrow{\varphi_1} L_0 \xrightarrow{\varphi_0} M \longrightarrow 0$$

where the filtration  $FL_1$  of  $L_1$  is constructed by setting  $d_{\text{fil}}(\varepsilon_k) = d_{\text{fil}}(\xi_{j_k})$ ,  $1 \leq k \leq m_1$ , and  $\varphi_1$  is defined by setting  $\varphi_1(\varepsilon_k) = \xi_{j_k}$ ,  $1 \leq k \leq m_1$ . If  $N_1 \neq 0$ , then, with the initial input data  $U = U_1$ , repeat Procedure 3 – Procedure 5 for  $N_1$  and so on.

By Theorem 5.6.3, a minimal filtered free resolution  $\mathcal{L}_\bullet$  of  $M$  gives rise to a minimal graded free resolution  $G(\mathcal{L}_\bullet)$  of  $G(M)$ . Since  $G(A) = K[\sigma(a_1), \dots, \sigma(a_n)]$  is a solvable polynomial algebra by Theorem 5.1.5, it follows from Theorem 4.4.1 that  $G(\mathcal{L}_\bullet)$  terminates at a certain step, i.e.,  $\text{Ker}G(\varphi_d) = 0$  for some  $d \leq n$ . But  $\text{Ker}G(\varphi_d) = G(\text{Ker}\varphi_d)$  by Proposition 5.5.1, where  $\text{Ker}\varphi_d$  has the filtration induced by  $FL_d$ , thereby  $G(\text{Ker}\varphi_d) = 0$ . Consequently  $\text{Ker}\varphi_d = 0$  since all filtration we are dealing with are separated, thereby a minimal finite filtered free resolution of length  $d \leq n$  is achieved for  $M$ .

## References

- [AF] F.W. Anderson and K.R. Fuller, *Rings and categories of Modules*. Springer-Verlag, 1974.
- [AL1] J. Apel and W. Lassner, An extension of Buchberger's algorithm and calculations in enveloping fields of Lie algebras. *J. Symbolic Comput.*, 6(1988), 361–370.

- [AL2] W. W. Adams and P. Loustau, *An Introduction to Gröbner Bases*. Graduate Studies in Mathematics, Vol. 3. American Mathematical Society, 1994.
- [AVV] M. J. Asensio, et al, A new algebraic approach to microlocalization of filtered rings. *Trans. Amer. math. Soc.*, 2(316)(1989), 537-555.
- [Ber1] R. Berger, The quantum Poincaré-Birkhoff-Witt theorem. *Comm. Math. Physics*, 143(1992), 215–234.
- [Ber2] G. Bergman, The diamond lemma for ring theory. *Adv. Math.*, 29(1978), 178–218.
- [Bu1] B. Buchberger, *Ein Algorithmus zum Auffinden der Basiselemente des Restklassenringes nach einem nulldimensionalen Polynomideal*. PhD thesis, University of Innsbruck, 1965.
- [Bu2] B. Buchberger, Gröbner bases: An algorithmic method in polynomial ideal theory. In: *Multidimensional Systems Theory* (Bose, N.K., ed.), Reidel Dordrecht, 1985, 184–232.
- [BW] T. Becker and V. Weispfenning, *Gröbner Bases*. Springer-Verlag, 1993.
- [CDNR] A. Capani, et al, Computing minimal finite free resolutions. *Journal of Pure and Applied Algebra*, (117& 118)(1997), 105 – 117.
- [DGPS] W. Decker, et al, SINGULAR 3-1-3 — A computer algebra system for polynomial computations. Available at [http://www.singular.uni-kl.de\(2011\)](http://www.singular.uni-kl.de(2011)).
- [Dir] P. A. M. Dirac, On quantum algebra. *Proc. Camb. Phil. Soc.*, 23(1926), 412–418.
- [Eis] D. Eisenbud, *Commutative Algebra with a View Toward to Algebraic Geometry*, GTM 150. Springer, New York, 1995.
- [EPS] D. Eisenbud, et al, *Non-commutative Gröbner bases for commutative algebras*. *Proc. Amer. Math. Soc.*, 126, 1998, pp. 687-691.
- [Fau] J.-C. Faugère. A new efficient algorithm for computing Gröbner bases without reduction to zero (F5). In: *proc. ISSAC'02*, ACM Press, New York, USA, 75-82, 2002.
- [Fröb] R. Fröberg, *An Introduction to Gröbner Bases*. Wiley, 1997.
- [Gal] A. Galligo, Some algorithmic questions on ideals of differential operators. *Proc. EUROCAL'85*, LNCS 204, 1985, 413–421.
- [Gol] E. S. Golod, Standard bases and homology, in: *Some Current*



- Trends in Algebra*. (Varna, 1986), Lecture Notes in Mathematics, 1352, Springer-Verlag, 1988, 88-95.
- [Gr] E. Green, *Noncommutative Gröbner Bases, A Computational and Theoretical Tool*. Lecture notes, Dec. 15, 1996. Available at [www.math.unl.edu/~shermiller2/hs/green2.ps](http://www.math.unl.edu/~shermiller2/hs/green2.ps)
  - [GV] J. Gago-Vargas, Bases for projective modules in  $A_n(k)$ . *Journal of Symbolic Computation*, 36(2003), 845C853.
  - [Hay] T. Hayashi,  $Q$ -analogues of Clifford and Weyl algebras-Spinor and oscillator representations of quantum enveloping algebras. *Comm. Math. Phys.*, 127(1990), 129-144.
  - [Hu] J. E. Humphreys, *Introduction to Lie Algebras and Representation Theory*. Springer, 1972.
  - [Jat] V.A. Jategaonkar, A multiplicative analogue of the Weyl algebra. *Comm. Alg.*, 12(1984), 1669-1688.
  - [JBS] A. Jannussis, et al, Remarks on the  $q$ -quantization. *Lett. Nuovo Cimento*, 30(1981), 123-127.
  - [Kr1] U. Krämer, Notes on Koszul algebras. 2010. Available at <http://www.maths.gla.ac.uk/~ukraehmer/connected.pdf>
  - [Kr2] H. Kredel, *Solvable Polynomial Rings*. Shaker-Verlag, 1993.
  - [KP] H. Kredel and M. Pesch, MAS, modula-2 algebra system. 1998. Available at <http://krum.rz.uni-mannheim.de/mas/>
  - [KR1] M. Kreuzer, L. Robbiano, *Computational Commutative Algebra 1*. Springer, 2000.
  - [KR2] M. Kreuzer, L. Robbiano, *Computational Commutative Algebra 2*. Springer, 2005.
  - [K-RW] A. Kandri-Rody and V. Weispfenning, Non-commutative Gröbner bases in algebras of solvable type. *J. Symbolic Comput.*, 9(1990), 1-26.
  - [Kur] M.V. Kuryshkin, Opérateurs quantiques généralisés de création et d'annihilation. *Ann. Fond. L. de Broglie*, 5(1980), 111-125.
  - [Lev] V. Levandovskyy, *Non-commutative Computer Algebra for Polynomial Algebra: Gröbner Bases, Applications and Implementation*. Ph.D. Thesis, TU Kaiserslautern, 2005.
  - [Li1] H. Li, *Noncommutative Gröbner Bases and Filtered-Graded Transfer*. Lecture Note in Mathematics, Vol. 1795, Springer-Verlag, 2002.
  - [Li2] H. Li, *Grobner Bases in Ring Theory*. World Scientific Publishing Co., 2011.

- [Li3] H. Li, On monoid graded local rings. *Journal of Pure and Applied Algebra*, 216(2012), 2697 – 2708.
- [Li4] H. Li, A Note on Solvable Polynomial Algebras. *Computer Science Journal of Moldova*, 1(64)(2014), 99–109. Available at arXiv:1212.5988 [math.RA].
- [Li5] H. Li, Computation of minimal graded free resolutions over  $\mathbb{N}$ -graded solvable polynomial algebras. Available at arXiv:1401.5206 [math.RA]
- [Li6] H. Li, Computation of minimal filtered free resolutions over  $\mathbb{N}$ -filtered solvable polynomial algebras. Available at arXiv:1401.5464 [math.RA]
- [Li7] H. Li, On Computation of Minimal Free Resolutions over Solvable Polynomial Algebras. (pp66) accepted by *Commentationes Mathematicae Universitatis Carolinae*, to appear.
- [LVO] H. Li and F. Van Oystaeyen, *Zariskian Filtrations*. *K-Monograph in Mathematics*, Vol.2. Kluwer Academic Publishers, 1996.
- [LW] H. Li and Y. Wu, Filtered-graded transfer of Gröbner basis computation in solvable polynomial algebras. *Communications in Algebra*, 1(28)(2000), 15–32.
- [Man] Yu.I. Manin, *Quantum Groups and Noncommutative Geometry*. Les Publ. du Centre de Recherches Math., Universite de Montreal, 1988.
- [Mor] T. Mora, An introduction to commutative and noncommutative Gröbner Bases, *Theoretic Computer Science*, 134(1994), 131–173.
- [MP] J.C. McConnell and J.J. Pettit, Crossed products and multiplicative analogues of Weyl algebras. *J. London Math. Soc.*, (2)38(1988), 47–55.
- [MR] J.C. McConnell and J.C. Robson, *Noncommutative Noetherian Rings*. Wiley-Interscience Publication, 1987.
- [NVO] C. Năstăsescu and F. Van Oystaeyen, *Graded ring theory*, Math. Library 28, North Holland, Amsterdam, 1982.
- [Ros] A.L. Rosenberg, *Noncommutative Algebraic Geometry and Representations of Quantized Algebras*. Kluwer Academic Publishers, 1995.
- [Rot] J.J. Rotman, *An introduction to homological algebra*. Academic Press, 1979.

- [Sch] F.O. Schreyer, *Die Berechnung von Syzygien mit dem verallgemeinerten Weierstrasschen Divisionsatz*. Diplomarbeit, Hamburg, 1980.
- [Sm] S.P. Smith, Quantum groups: an introduction and survey for ring theorists. in: *Noncommutative rings*, S. Montgomery and L. Small eds., MSRI Publ. 24(1992), Springer-Verlag, New York, 131–178.
- [SWMZ] Y. Sun, et al, A signature-based algorithm for computing Gröbner bases in solvable polynomial algebras. In: *Proc. IS-SAC'12*, ACM Press, 351–358, 2012.
- [Uf] V. Ufnarovski, Introduction to noncommutative Gröbner basis theory. in: *Gröbner Bases and Applications* (Linz, 1998), London Math. Soc. Lecture Notes Ser., 251, Cambridge Univ. Press, Cambridge, 1998, 259–280.
- [Wal] R. Wallisser, Rationale approximation der  $q$ -analogues der exponentialfunktion und Irrationalitätsaussagen für diese Funktion. *Arch. Math.*, 44(1985), 59–64.
- [Wik1] Decision problem (Solvable problem). Available at [https://en.wikipedia.org/wiki/Solvable\\_problem](https://en.wikipedia.org/wiki/Solvable_problem)
- [Wik2] Confluence (abstract rewriting). Available at [https://en.wikipedia.org/wiki/Confluence\\_\(term\\_rewriting\)](https://en.wikipedia.org/wiki/Confluence_(term_rewriting))